Mathematical Representation at the Interface of Body and Culture

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CHAPTER 7

HOW DO YOU KNOW WHICH WAY THE ARROWS GO?

The Emergence and Brokering of a Classroom Mathematics Practice

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In this chapter we analyze how a particularly rich and complex inscription known as a bifurcation diagram emerged in an undergraduate differential equations class. We frame the analysis in terms of the construct of a classroom mathematics practice and demonstrate how the brokering moves of the teacher and some students functioned as a mechanism for the initiation and ongoing growth of this practice. Our analysis contributes more broadly to contemporary accounts of how creating, using, and interpreting inscriptions is a social process that reflects the evolving culture of the classroom community, as well as the culture of the broader mathematical community.

The chapter is divided into three main sections. The background section provides a brief survey of the literature on representations, followed by a discussion of what is meant by a classroom mathematics practice and by brokering. The results and discussion section consists of four parts that

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chronologically analyze students' reasoning over the course of two class sessions. In the conclusion section, we detail the three broad categories of broker moves that are identified through our reflection on the various broker moves that serve as a mechanism for the emergence of a bifurcation diagram.

COMMUNITIES OF MATHEMATICAL PRACTICE

The emerging classroom mathematics practice that we analyze in this chapter centers on students' (re)invention of a fairly complex representation known by experts as a bifurcation diagram. To situate our analysis within the vast literature on representations, we begin with a brief review of some early work that was dominated by cognitive framings and then point to more recent work informed by contemporary social and cultural perspectives.

A Social Practice Perspective on Mathematics

In a move away from the traditional representational view of mind approach, Roth and McGinn (1998) replace the problematic notion of representation with the term inscription, which refers to "signs that are materially embodied in some medium, such as paper or computer monitors" (p. 37). Examples of inscriptions are tables, lists, diagrams, graphs, equations, and spreadsheets. Because of their material embodiment, inscriptions are accessible by all learners, as opposed to mental representations, which are inaccessible to all members of a learning community except the particular individual whose mental representations might be under consideration. Roth and McGinn, therefore, assert that by focusing on the development, transformation, and use of graphical inscriptions in a social setting rather than on the activity of an individual mind, we are presented with a richer framing through which to understand learners and graphical symbolism. Our analysis in this chapter aims to contribute in this direction.

This shift in focus coincides with an increased interest in theoretical and pragmatic accounts of learning and teaching from cultural and sociological points of view (Wenger, 1998). A common thread among this more recent work is a focus on how communities of learners participate in and develop various classroom mathematics practices that tend to become normative with respect to how students in a classroom community reason, symbolize, and argue. Such practices are established through patterns of interaction center on specific forms of mathematical activity. For example,

one of six practices emerging in an undergraduate mathematics course in differential equations was referred as the "creating and structuring a slope field practice" (Stephan & Rasmussen, 2002). This particular classroom mathematics practice entailed three specific ways of reasoning: detailing the way in which slopes change over time, justifying why slopes are invariant horizontally for autonomous differential equations, and imagining an infinite number of tangent vectors when only finitely many are visible. As this example illustrates, these researchers characterize a classroom mathematics practice in terms of a *cluster* of related forms of activity. This characterization grows out of yet differs from earlier characterizations put forth by Cobb and Yackel (1996) in that the early characterization of a classroom mathematics practice involved one form of mathematical activity, rather than a cluster. Classroom mathematics practices typically evolve over several days and even weeks. As such, classroom mathematics practices are not pre-established forms of activity into which students are inducted, but rather emerge as participants interact. Hence, different classrooms using the same curriculum can develop a somewhat different collection of classroom mathematics practices with somewhat different forms of activity.

The nature of classroom mathematics practices involves two additional characteristics. First, classroom mathematics practices can be established in a non-sequential temporal order. In a previous analysis that documented the constitution of classroom mathematics practices in a first-grade mathematics class (Stephan, Bowers, Cobb, & Gravemeijer, 2003), the various practices proceeded in a more or less sequential fashion: the initiation and constitution of the first practice preceded the initiation and constitution of the second practice in time. Such temporal orderliness is not always the case (Stephan & Rasmussen, 2002). Second, classroom mathematics practices can emerge in a non-sequential structure. That is, one or more of the specific forms of activity that make up one practice can also be part of a different practice. For example, in creating and structuring a slope field practice, the form of activity referred to as justifying why slopes are invariant horizontally for autonomous differential equations might very well appear in an entirely different classroom mathematics practice. The analysis presented in this chapter takes this structural overlap even further by suggesting that an entire classroom mathematics practice can become embedded in another practice.

Evolving classroom cultures tend to be mediated by the broader culture of the discipline of mathematics. This is because the goals and values of the classroom participants, as well as the curricular materials, are inseparable from the discipline itself. Teachers and texts do not exist in isolation from the practices of the broader field of mathematics. The practices of this broader culture can be characterized in terms of the more generic

activities of defining, algorithmatizing, proving, modeling, and symbolizing (Rasmussen, Zandieh, King, & Teppo, 2005). The classroom mathematics practice that we analyze in this chapter is a particular manifestation of the discipline practice of symbolizing. Mathematicians routinely create, interpret and use inscriptions to solve problems, communicate with colleagues, convince others, and so forth. Thus, particular classrooms, especially those that are inquiry-oriented with the intent of engaging students in the authentic practice of mathematics, might engage learners in one or more of these broader discipline practices. In this way, students' participation in a local classroom mathematics practice is a way in which they are enculturated into the cultural practices of mathematics.

Brokering and the Emergence of Mathematics Practices

In this chapter we offer insights into the mechanisms by which classroom mathematics practices are initiated and begin to take hold. To accomplish this we analyze in depth two successive class sessions. We demonstrate how brokering functions as a mechanism that supports the initiation and emergence of one particular classroom mathematics practice in which students engage in symbolizing. Our use of the term brokering draws on the work of Wenger (1998). Brokering is a mechanism that influences the degree of continuity between communities. In our analysis, we consider three different communities: the broader mathematical community, the local classroom community, and the various small groups that make up the local classroom community. As we detail in this chapter, the brokers in these communities are the teacher and specific students in the class. A broker is someone who has membership status in more than one community. For example, in our case the teacher is a legitimate peripheral member of the broader mathematics community, a full member of the classroom community, and a peripheral member of each of the small groups that make up the classroom community.

The job of brokering is complex (Wenger, 1998). Brokers facilitate the translation, coordination, and alignment of perspectives between communities. Brokers must be able to mobilize attention and address different points of view. Brokers are unique in that they are "able to make new connections across communities of practice, enable coordination, and—if they are good brokers—open new possibilities for meaning" (p. 109). Wenger also argues that the job of brokering requires the ability to "cause learning" by introducing into one community elements of practice from a different community (p. 109). For example, the teacher as broker might introduce formal terminology from the discipline of mathematics into the

classroom community. On the other hand, particular students who are adept at certain forms of activity that make up a classroom mathematics practice might act as brokers to facilitate others in the class as they move from periphery to more central participants in this practice. This latter example is brokering between different communities within the same classroom. In other words, learning is evidenced by the initiation and evolution of classroom mathematics practices, and it is toward this end that we now turn.

A CASE STUDY OF BROKERING

In this section we detail the initiation of a classroom mathematics practice (CMP) that involved the creation and interpretation of a bifurcation diagram and the brokering moves that functioned as a mechanism for the emergence and appropriation of this practice. We refer to this particular CMP as the "summarizing the changing structure of the solution space practice." By structure of the solution space, we mean the characteristics of the collection of functions that satisfy the differential equation. These characteristics include the number, location, and type (sink, source, or node) of equilibrium solutions, the concavity of solution graphs, and the horizontal invariance of graphs of solution functions to autonomous differential equations.[1] In our analysis, we demonstrate how the teacher, together with the students from one particular small group, functioned as brokers for others in the class to interpret, understand, and participate in this particular practice.

Data for this analysis comes from a 15-week classroom teaching experiment conducted in an undergraduate differential equations course. Data sources consisted of daily classroom video recordings from two cameras, video recordings of student interviews, and copies of student work. The teaching experiment was conducted as part of a larger research program aimed at developing an inquiry-oriented, research-based instructional approach in undergraduate mathematics.

Our analysis focuses on data collected from two consecutive classroom sessions. We find that this particular CMP entails five related ways in which students worked with various inscriptions and related symbolic equations. In particular, the summarizing the changing structure of the solution space practice consisted of five forms of mathematical activity (Figure 7.1). The forms of activity figure prominently in the analyses presented in the subsequent discussion. We determined these five forms of activity retrospectively through repeated examination of the data, but present them here prospectively to facilitate organization and discussion of results.

(a) Reasoning with single dy/dt versus y graph to ascertain the structure of the solution space.

(b) Reasoning with discretely or continuoulsy changing dy/dt versus y graphs to ascertain the changing structure space.

(c) Solving the equation $dy/dt = 0$ and interpreting the result in terms of the equilibrium values as a function of some parameter.

(d) Plotting the equilibrium solutions (values) as they vary as a function of the paramenter.

(e) Re-presenting the structure of the solution space with phase lines embedded in the equilibrium value versus parameter graph.

Figure 7.1. Summarizing the changing structure of the solution space practice

We present our study in four parts. In each of the four parts we emphasize the brokering moves that function as a mechanism for the emergence of the summarizing the changing structure of the solution space practice. In addition, in the first part we describe the initial task that eventually led to some students creating a bifurcation diagram and detail students' initial work on it. In the second part we detail the presentation of Brady's small group and point to ways in which their presentation was similar to or different from the forms of mathematical activity that constitute the summarizing the changing structure of the solution space practice. In the third part we describe how Kenneth and Lorenzo's small group presentation essentially reinvented a bifurcation diagram and how their presentation relates to the various forms of activity that constitute the changing structure of the solution space practice. In the final part we return to Brady's group to detail their work on a new task and their transition to becoming more central participants in the summarizing the changing structure of the solution space practice.

Part 1: Introduction of and Initial Work on the Fish.com Task

In this section we detail the initiation and start of a problem that eventually led to one group of students creating what an expert would recognize as a bifurcation diagram. To emphasize, the bifurcation diagram these students invented was the product of their own creative activity and not something that was appropriated from a textbook, a teacher, or some other resource. Moreover, the creation, interpretation, and further

rendering of this inscription entailed a number of different forms of mathematical activity (Figure 7.1). It is these various forms of mathematical activity that constitute the summarizing of the changing structure of the solution space practice.

In the first of the two sessions analyzed here, the teacher introduced the "fish.com" problem (Figure 7.2). After a brief discussion of the problem statement, the teacher discussed with students the need to modify the differential equation $\frac{dP}{dt} = 2P\left(1 - \frac{P}{25}\right)$ in some way because the given differential equation does not take into account harvesting by the local public. Before trying to decide on an appropriate modification, however, the teacher requested that students work in their small groups to first fully understand what the rate of change equation $\frac{dP}{dt} = 2P\left(1 - \frac{P}{25}\right)$ predicts for the fish population for various initial conditions. After completing their analysis, the teacher invited two students from different small groups to present their findings. Figure 7.3 shows the different inscriptions that these students used in their analysis.

In Figure 7.3a, Roy presents a sketch of a slope field, which he said was generated using his graphing calculator, and the corresponding flow line (phase line) to the right of the slope field. The features of the graphs that Roy chose to discuss included how to draw the flow line from the slope field, the existence of equilibrium solutions at $P = 0$ and $P = 25$, and the conclusion that solutions with positive initial conditions "will all just kind of go towards 25." Roy's carefully drawn slope field strongly suggests horizontal invariance in the tangent vectors and the concavity for various solutions. Although negative P values do not make sense for this situation, students in this class had been accustomed to analyzing the full range of solutions for the sake of mathematical completeness. Roy's graphs reflected such analyses.

A scientist at fish hatchery has previously demonstrated that the rate of change equation $\frac{dP}{dt} = 2P\left(1 - \frac{P}{25}\right)$ is a reasonable model for predicting the number of fish that the hatchery can expect to find in their pond.

Recently, the hatchery was bought out by fish.com and the new owners are planning to allow the public to catch fish at the hatchery (for a fee of course). The new owners need to decide how many fish per year they should allow to be harvested. Prepare a report for the new owners that illustrates the implications that various choices for harvesting will have on future fish populations.

Figure 7.2. Fish.com problem.

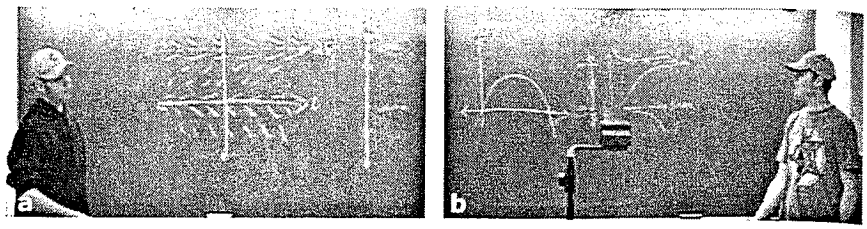


Figure 7.3. Student analyses of $\frac{dP}{dt} = 2P\left(1 - \frac{P}{25}\right)$.

In Figure 7.3b, Kurt presented similar conclusions about the solution functions $P(t)$, but his work utilizes a graph of dP/dt versus P rather than a slope field. Based on what happened during the two lessons, it appears that interpreting such graphs to ascertain the structure of the solution space was in and of itself a distinct CMP. This finding is consistent with other findings in a different, but similarly taught differential equations class (Stephan & Rasmussen, 2002). In reference to Figure 7.1, this distinct practice becomes embedded in the summarizing of the changing structure of the solution space practice. Because use of such graphs will figure prominently in the subsequent sections, we provide transcript and commentary on how Kurt (and his group) interpreted and reasoned with such inscriptions.

In his presentation, Kurt pointed back and forth between the graph of dP/dt and the flow line immediately to the right of the parabola shown in Figure 7.3b.

> Kevin: Greater than 25 [points to the graph of dP/dt versus P where $P > 25$] you're going to have a negative [points to the portion of the flow line above $P = 25$ with a downward pointing arrow]. Between 0 and 25 [points back to the graph of dP/dt versus P where $0 < P < 25$] you're in the positive [points back to the portion of the flow line with an upward pointing arrow], when it's less than zero [points back to the graph of dP/dt versus P where $P < 0$] you're going to have a negative [points back to the flow line where $P < 0$].

Kurt's interpretation of the dP/dt versus P graph focused on the regions in which the sign of dP/dt is positive or negative. For example, for P between 0 and 25, Kurt stated that "you're in the positive," meaning that the graph is *in* the region above the P-axis. Similarly, when the graph of dP/dt is below the P-axis the rate of change is negative and hence the flow line would point downward to indicate decreasing solution functions. This is consistent with how students in the class interpreted other graphs of

autonomous differential equations. Next, Kurt explained how he then drew representative graphs of the functions that solve the differential equation.

> Kurt: And from there you can just draw like basic equations, generalizations of it [points to the flow line between 0 and 25]. So if you start anywhere between 0 and 25, it's [points to the graph of P versus t and begins to trace out the solution graph as he talks] going to increase slowly by zero and then more rapidly in the middle and then slower again at 25. It's going to decrease to 25 [tracing out the graph of P versus t above 25]. Same thing for the opposite [i.e., when P is less than 0] but you can't have a negative fish population, so it's all going to go towards 25.

Although Kurt did not elaborate on what he meant by "basic equations," students' prior work in this class suggests that he might have meant that the graphs he drew were "basic" in the sense that other solution graphs could be obtained from these simply by shifting the graphs drawn left or right. This result follows from the fact that the differential equation is autonomous, and is consistent with Roy's slope field that shows tangent vectors with a horizontal invariance to their slopes. Kurt also noted that, except for negative fish populations, "it's all going towards 25." In other words, the equilibrium solution of 25 attracts nearby solutions. Students in this class referred to such equilibrium solutions as a "sink."

Kurt's explanation involved a significant amount of pointing back and forth between the various inscriptions, which implicitly brought forth connections between the inscriptions. Neither the teacher nor other students requested additional explanation or clarification for how he came to his conclusions and for why these conclusions were valid. The teacher even provided ample opportunity for students to ask Kurt questions. For example, after Kurt finished explaining, the teacher commented as follows:

> Teacher: So both Roy and Kurt it seems to me have the same conclusions but slightly different ways of going about organizing or getting that information. Good. Ok, let's get, are there any questions on this? Folks want to ask them questions, clarifying questions, or bring up some more points they want to be more picky on? [14 second pause] Everyone ok then? Ok, ok let's go to the—thanks, guys—so, so the next part of the problem.

In inquiry-oriented classrooms such as this one, the lack of or dropping off of challenges or requests for justification is a sign that particular ideas and interpretations are functioning "as if" the classroom community shared them (Rasmussen & Stephan, 2008). In this particular class, stu-

dents routinely indicated disagreement or asked questions when in doubt. However, they did not do so in this case, which suggests that the ideas related to Kurt's presentation functioned as if they were shared by others in the class. In other words, it appears that reasoning with a graph of dy/dt versus y had become relatively routine for these students. Indeed, Kurt's use of a dP/dt versus P graph was not the first time that students had used such an inscription to glean information about the structure of the solution space. For example, in a previous class session students had been given a graph of an autonomous differential equation dN/dt as a function N and had used this graph to ascertain the structure of the solution space.

In terms of the reasoning about the changing structure of the solution space practice, Kurt's interpretation and use of the graph of dP/dt versus P is listed as aspect "a" in Figure 7.1. As stated earlier, we see this particular form of activity as a distinct CMP. While a full accounting of the forms of activity that constitute "reasoning with a single dy/dt versus y graph to ascertain the structure of the solution space" is beyond the scope of this chapter, we surmise from Kurt's presentation that this CMP includes the following forms of activity: (a) interpreting the y axis intercepts of a dy/dt versus y graph as equilibrium solutions; (b) determining whether solution functions are increasing or decreasing by graphically examining the sign of the derivative; (c) labeling the equilibrium solutions (e.g., sink or source) by the behavior of nearby solutions; and (d) making connections between a graph of dy/dt versus y, a phase line, and solution graphs in the y versus t plane. All of these activities were evident in Kurt's presentation, and are part of Parts 3 and 4 of this section.

After Roy and Kurt's presentation, the teacher then requested that students try to modify the rate of change of equation $\frac{dP}{dt} = 2P\left(1 - \frac{P}{25}\right)$ so that it would account for the new owners' decision to allow the public to catch fish. For approximately 6 minutes students brainstormed various ways to modify the differential equation in their small groups. This was followed by a 14-minute whole class discussion of students' ideas and rationales. This discussion served to bring out ideas from the small groups and to cultivate new ideas that emerged for some students as their classmates explained their thinking. The different ideas that were discussed in whole class were to subtract from the given differential equation the following: a constant k, some function of time $h(t)$, the term $k\left(1 - \frac{P}{12.5}\right)$, and a fraction of the current population, for example $\frac{1}{3}p$. The rationales for these ideas that students gave were quite sophisticated, and the teacher noted the many sensible ideas for modifying the differential equation. In the end, however, the teacher decided that the class was

to use the following modified differential equation: $\frac{dP}{dt} = 2P\left(1 - \frac{P}{25}\right) - k$, where k represents a constant annual harvesting rate. The teacher justified this choice to students as follows: "So the reason I'm doing this [choosing the $-k$ modification] is because I didn't want 10 different groups having different differential equations because then we don't have a common basis to talk about our analyses." Although the teacher did not mention this to the class, we note that this particular modification, in comparison to the other suggested modification, lends itself more readily to discretely or continuously changing the original dy/dt versus y graph.

Teacher Broker Moves

In the previous paragraphs we outline the various forms of activity that constitute the summarizing the changing structure of the solution space practice and begin detailing students' mathematical reasoning related to these forms of activity. Next, we focus our attention on the teacher to highlight the unique broker moves as they played out in Part 1. Even in the relatively short amount of class time that transpired in Part 1, the teacher played a number of distinct roles as a broker between the classroom community and the mathematical community, and between the classroom community as a whole and individual small groups.

First, the teacher played a critical role as broker between the classroom community and the broader mathematical community in terms of task selection and the way in which he engaged students in the task. Specifically, it was the teacher who selected the fish.com task because it had the potential for student creation and reinvention. In other words, it is the teacher who is in a position to recognize characteristics of tasks that are likely to be productive for enabling newcomers to be more central participants in the discipline of mathematics, such as modeling and symbolizing. For example, the fish.com problem was recognized by the teacher as a task he could use to engage students in modeling by revising a differential equation to fit new assumptions and analyzing this new model. The fish.com problem, as used by the teacher, also engaged students in symbolizing by virtue of the fact that the task invited them to develop "a report for the new owners that illustrate the implications that various choices for harvesting will have on future fish populations." The teacher did not prescribe what form this report was to take and hence allowed students to develop their own inscriptions for the owners. At the same time the teacher opened up the task in terms of not prescribing methods for analysis or ways to present findings, the teacher also set constraints and boundaries on the task. In particular, the teacher emphasized to the class

that their job was not to tell the owners what harvesting rate to use, but rather to provide information to the owners so that they could make an informed decision. The teacher also constrained presentations to a single overhead transparency to encourage students to figure out a way to consolidate their analysis. As such, the teacher's task selection and the way in which he engaged students in the task provided the opportunity for him to act as a broker between the discipline of mathematics and the way in which the classroom community engaged in these activities.

Second, the teacher played a critical role as broker between the classroom community and different small groups within this community. One particularly noteworthy brokering move occurred when the teacher invited two different students, Roy and Kurt, to share their analysis of the original differential equation with the whole class and others to comment or ask questions about the presentations. The significance of this broker move is that it resulted in situating the new problem within previously established practices in a way that did not privilege particular analysis techniques nor prescribe how to proceed. For example, after Kurt's presentation the teacher commented, "So both Roy and Kurt it seems to me have the same conclusions but slightly different ways of going about organizing or getting that information. Good." It also encouraged ownership of analyses, positioned students as capable of making progress, and facilitated connections across approaches.

All of the brokering moves that occurred in Part 1 served the purpose of creating the background that allowed for the emergence of the summarizing the changing structure of the solution space practice. Moreover, these brokering moves helped create a situation for subsequent brokering moves by the teacher for others in the class to more fully participate in this practice. In Parts 2–4 below we fully develop these points.

Part 2: Brady's Presentation of the Fish.com Task

In this part, we discuss and analyze Brady's presentation of his group'swork on the modified differential equation $\frac{dP}{dt} = 2P\left(1 - \frac{P}{25}\right) - k$ in the fish.com task. Brady's presentation relied heavily on reasoning with a table of harvesting rates and population values, and he made no explicit references to graphs of dP/dt versus P. He was able to engage in conversation with the class about his group's ideas, explain examples as well as connect to mathematical concepts not initially mentioned in his presentation. In Episode 2a, we detail Brady's presentation of his group's work. In Episode 2b, we discuss various ways in which the teacher and another student

served as brokers between Brady and the rest of the class as they facilitated the clarification of Brady's approach as well as the connection to previously established mathematical ideas. We conclude this section with a discussion in which we point to ways in which Brady's presentation was similar to or different from the forms of mathematical activity that constitute the summarizing of the changing structure of the solution space practice. In particular, we argue that Brady's presentation offered a tabular way to summarize the changing structure of the solution space that utilized reasoning with the *dy/dt* equation in a way that was more algebraic and numeric, in comparison to Kurt's presentation in Part 1 and to what we will see in Part 3. In addition, we claim that Brady's presentation provides evidence that he and his group were poised to participate in the emerging practice, which we will detail in Part 4.

Episode 2a: Brady's Presentation

Brady was the second person to present a small group's work on the modified differential equation $\frac{dP}{dt} = 2P\left(1 - \frac{P}{25}\right) - k$ in the fish.com task. As we see from Brady's presentation, the table served as the group's main tool for summarizing the situation and making decisions about harvesting rates in the required 1-page form (Figure 7.4). The table's left-most column contained the values that Brady and his group used for the harvesting rate H (his group used "H" instead of "k" as the variable for the harvesting rate parameter). Notice that these H-values are integers, ranging from one to twelve. The other two columns, labeled "POP MAX" and "POP MIN," contained the values that bound the range of the maximum and minimum populations that are associated with each particular harvesting rate that yield a positive rate of change.

Brady began his presentation by stating his group's initial thinking that it would be useful or "ideal" to maintain a positive growth rate.

> Brady: Okay, um, when we did ours, um, our idea was that since you have a holding factor of basically twenty-five, if you go over twenty-five you automatically start going down. So we said that it was ideal to keep your population between zero and twenty-five. So we kind of assumed that.

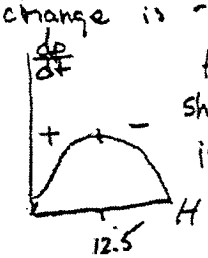
Brady then detailed how they used the equation, case by case for integer H-values between one and twelve, to fill in the POP MAX and POP MIN columns in their table: "Then we used our [differential] equation and we found that we had a max population and a minimum population for each H-value. So for every H-value if you had a population between this range, that you could go, pulling this many fish." Brady did not explain how he and his group used the differential equation to find the

H	POP_{MAX}	POP_{MIN}
1	24.48	.51
2	23.96	1.04
3	23.40	1.60
4	22.81	2.19
5	22.18	2.82
6	21.51	3.49
7	20.79	4.21
8	20.00	5.00
9	19.11	5.89
10	18.09	6.91
11	16.83	8.17
12	15.00	10.00

For Any picked H-value if the intial pop stays between Pop_{MAX} and Pop_{MIN} than you will have a positive growth rate.

In this chart we are showing the range in which you can pull fish and still have a positive growth rate. when the rate is going down than owners are losing money becau the lake will be empty with a few years

full is 12.5 but the first year rate of change is -12.5. $\frac{dp}{dt} = 2p\left(1-\frac{p}{25}\right)-H$



example. The largest H to

Here this graph supports are chart showing that the maxium harvest is 12 fish (since can't have 12½ fis

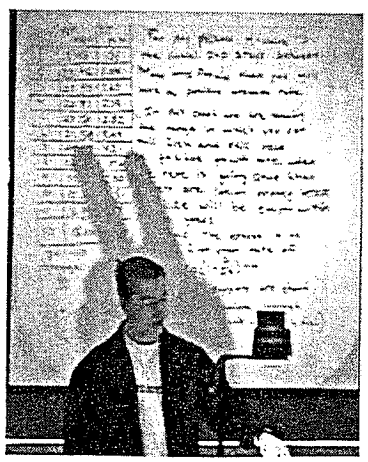
Figure 7.4. The overhead for Brady's group.

corresponding population values for each H-value. However, because the table has each population value given to two decimal places (Figure 7.4), we strongly suspect that the group solved for the values of P that were solutions to $2P\left(1-\frac{P}{25}\right)-H=0$ for each of the table's given H values.

Next, Brady illustrated how to interpret their table by taking the class through an example from the table. In particular, Brady asked the class to imagine that their pond had an initial population of twenty-one units of fish. He then found where twenty-one would be situated in the range of values given in the POP MAX and POP MIN columns and concluded that the chart's corresponding H-value of six (Figure 7.5) would be the best amount of fish to pull.

Brady: Like, uh, if you were gonna, if you had a population of 21, then, uh, you could say, okay, well 21 is between, uh, 21.5 one and 3.49, so your best *H* to hold would be 6. And the whole point of these numbers are, is, between these two numbers, when you pull that many fish, then you're going to have a positive rate of change. So you'll have an increase in the fish population, and we thought that, uh, an increase in the fish population here was an important thing for a fish company, because then they wouldn't have to restock their fish.

Brady's rationale for the "best *H*" corresponding to an initial population of 21 hinged on the fact that the rate of change between POP MIN and POP MAX is positive and hence, as Brady made explicit, "you'll have an increase in the fish population." Neither the teacher nor other students asked for clarification on how Brady and his group determined that the rate of change was positive. Perhaps he and his group determined the sign of dP/dt by inserting specific P-values into the differential equation, or perhaps they created slope fields similar to what Roy did (Figure 7.3a). Although we cannot be certain how they determined the sign of dP/dt, we think that it is highly unlikely that Brady and his group used a graph of dP/dt versus P for this purpose. Evidence for this interpretation comes from the fact that although a dP/dt versus P graph appears in the bottom left corner of their overhead (Figure 7.4), this graph was never mentioned or discussed by Brady. Moreover, the axes on the graph are incorrectly labeled. Thus, it is unlikely that Brady's group used this graph as a reasoning tool. Further evidence for this claim are detailed in Part 4.



H	rate MAX	rate MIN
1	24.48	.51
2	23.96	1.04
3	23.40	1.60
4	22.81	2.19
5	22.18	2.82
6	21.51	3.49
7	20.79	4.21
8	20.00	5.00
9	19.11	5.89

Figure 7.5. Brady explains how to use their chart.

Brady's explanation of how to interpret their table indicates his attention to the structure of the solution space. Recall that by "structure of the solution space," we mean the characteristics of the collection of functions that satisfy a differential equation, which includes the number, location, and type (sink, source, or node) of equilibrium solutions. For example, Brady used an initial population of 21 as a generic example to explain how to read their table. Using language closely tied to the context, his explanation essentially detailed how POP MAX functions as a sink and the POP MIN functions as source, thus indicating the structure of the solution space. Brady's ability to reason about the structure of the solution space will be revisited in the discussion, where we discuss how Brady's presentation relates to the summarizing the changing structure of the solution space practice.

Episode 2b: Teacher and Student Broker Moves

A few moments after Brady's explanation, the teacher—who was sitting down in a desk among the other students in the class—prompted Brady to explain his group's table a bit more.

> Teacher: Brady, let's pretend that I'm an owner and I make a lot of money, and I'm an executive, but I'm not so sophisticated with tables and stuff. So I need some help understanding that table that you've got up there.
>
> Brady: Oh, okay. Alright, for each *H*-value, that's how many fish you can pull [he points at the table] and for each one, if you go across, the population max is—and the population min—that is the range of populations; that's, uh, so it's the highest amount of fish and the lowest amount of fish that you can have in your lake, pull this many fish out, and still get a positive rate of change. I don't know if that helps.

The teacher phrased his request for clarification in such a way that kept the tone of the original phrasing of the task yet prompted Brady to delve further into his role of explaining his group's ideas to the classroom. The teacher's phrasing, as well as positioning himself amongst the other students in the class, both served as indicators of the high value the teacher placed on particular norms in the class. First, we see the teacher placing himself as a member of the class rather than as the outside expert and authority, emphasizing the legitimacy of students as participants rather than spectators in the learning process. We also see this an example of the teacher serving as a broker between the understandings of one community—Brady's small group—and another community—that of the classroom as a whole. By asking Brady to explain again for someone who may not be "so sophisticated with tables and stuff," the teacher encouraged the development of understanding each other's ideas, a norm that had been in development since the semester began. Brady concluded with, "I don't

know if that helps," which implies that he realized his explanation was supposed to serve the purpose of further clarifying his group's approach. This claim is supported by the video, which shows Brady looking around the room, rather than only at the teacher, who initially requested more explanation.

After Brady's explanation, which ended with "I don't know if that helps," the teacher followed up by acting in accordance with the norm that students listen to and respond to their classmates' questions or requests.

Teacher: [7 second pause] So that's a question to you all—does that explain what his table is? If you're the owners, are you understanding what he's, what information he's provided? Nathan—kind of?

Nathan: Well, if I didn't know anything about math, you know, like, then probably not. But I understand what he's saying.

Teacher: Summarize for us, Nathan, your interpretation of the information he's presenting.

Nathan: He's saying that the max and min values are like the equilibrium points, where if you're in that range and you pull that amount of fish, it will be increasing towards the max equilibrium point.

Brady: Yeah, and if you are exactly on the number then you would have a zero rate of change.

Nathan: Right, it'll just stay there.

The teacher waited for seven seconds before he spoke, which allowed sufficient time for students to contribute their response if they had one. When no class member spoke, the teacher did not comment on the potential effectiveness of Brady's explanation; rather, he directly requested that the students respond as if they were the owners of fish.com. This short exchange shows the benefit of the teacher acting in accordance with the norm that students listen and respond to their classmates' ideas. In particular, as a result of the teacher's actions, Nathan in essence acted as a broker to connect Brady's group terminology of POP MIN and POP MAX to the conventional terms of equilibrium points. When asked to summarize his interpretation of Brady's work, Nathan stated, "The max and min values are like the equilibrium points." Brady immediately agreed and added more interpretation about what it meant to have a zero rate of change. Brady's agreement indicates that Nathan had interpreted the results as he had intended.

After a bit of discussion about the contextual sensibility of particular population values and their corresponding H-values, the teacher directed Brady's attention back to Nathan's assertion that POP MIN and POP MAX were equilibrium solutions.

Teacher: Oh, does it, is that [POP MIN of 1.04] an equilibrium? I think I heard Nathan say—

Brady: I think it is an equilibrium point.

Teacher: Are those population min and population max, are those supposed to be equilibrium solutions?

Brady: Yeah, yeah. If your population is 1.4 [sic 1.04] and you pull 2 fish a year, then your population will stay at 1.04.

We see the teacher's statement "Is that an equilibrium? I think I heard Nathan say—" as serving two purposes. First, explicitly attributing ideas to students enforces a sense of ownership of ideas among the students, which in turn helps students develop a view of themselves as capable thinkers and doers of mathematics. Second, the teacher called attention to Nathan's role of brokering in this situation after the latter had previously "translated" Brady's explanation into the notion of equilibrium points. This was a well-established idea by this stage of the semester for this class. Finally, the teacher himself served as a broker here as well, choosing to further emphasize the association between what Brady was describing contextually and a previous mathematical idea.

Connections to the Mathematical Activity of the Emerging Practice

Brady's presentation suggested a tabular way to summarize the changing structure of the solution space that utilized reasoning with the *dP/dt* equation in a way that was more algebraic and numeric, in comparison to the graph-oriented reasoning we see in Kurt's presentation in Part 1 and in what we see in Part 3. That is, Brady's use of the *dP/dt* equation constitutes a separate but complementary mathematical activity than those listed as aspects "a" and "b" (Figure 7.1). In Part 1 we see Kurt reason with the *dP/dt* versus *P* graph to ascertain the structure of the solution space (aspect a) of the emerging practice, and in Part 3 below we see how another group presentation utilized *dP/dt* versus *P* graphs with varying *H*-values in order to ascertain the changing structure of the solution space, aspect "b" of the emerging practice. Brady and his group's table-based reasoning, while different than either of the aforementioned graph-related activities, provided both affordances and limitations for becoming more central participants in the emerging summarizing the changing structure of the solution space practice. Here we detail what affordances this way of thinking provided Brady and his group, and in Part 4 we return to what limitations they encountered by not reasoning with the *dP/dt* versus *P* graphs.

Brady's group did not appear to utilize to any large degree a graph of *dP/dt* versus *P*. Hence we see little evidence that they engaged in the forms of activity a) or b) from Figure 7.1, both of which capitalize on graphs of

dy/dt versus *y*. However, the group's way of reasoning, even without the graphs, aligned them to some degree with the underlying reasoning behind certain aspects of the emerging practice. Here we suggest that the group's table-based reasoning aligns them, at least partially, with the reasoning that is manifested in graphical form in activities "c," "d," and "e" of the emerging practice (Figure 7.1).

When explaining his group's approach, Brady stated in Episode 2a, "Then we used our equation and we found that we had a max population and a minimum population for each *H*-value." Although Brady did not explicitly use the terms "equilibrium point" or "equilibrium solution" in his presentation, he quickly agreed with both Nathan and the teacher when they used one of these terms in place of "population max" or "population min." The above quote, along with Brady's quick agreement with Nathan and the teacher about the terminology, are evidence that the tabular-based reasoning exhibited by Brady's group was well-aligned with the graph-oriented reasoning elicited in aspect "c" of the emerging practice. In addition, this is also true in regard to aspect "d." Brady and his group "plotted" the information in a table rather than in a graphical format. Although Brady's table presents only partial information about the relationship between the parameter and equilibrium solutions, the underlying reasoning behind this tabular presentation was very well-aligned with the reasoning that is manifested graphically in aspect "d" of the emerging practice.

Finally, Brady's presentation shows evidence that his group was in position to participate in the form of activity "e." Although Brady's group did not draw or discuss phase lines, his explanation of how to interpret their table included much of the same information that a graph of the equilibrium values versus parameter with phase lines embedded would have offered. Brady had said, "Alright, for each *H*-value ... it's the highest amount of fish and the lowest amount of fish that you can have in your lake, pull this many fish out, and still get a positive rate of change." The group therefore was already reasoning, at least in part, about the changing structure of the solution space. This information, provided verbally through Brady's explanation, provides essentially the same information that would be given by embedding phase lines in the equilibrium value versus parameter graph.

In conclusion, Brady's group members were in a position that afforded them the ability to become legitimate peripheral participants in the summarizing of the changing structure of the solution space practice, which was fully introduced later in the class period through Lorenzo and Kenneth's presentation (see Part 3 below). Furthermore, Brady's group's lack of reasoning with *dP/dt* versus *P* graphs served as a constraint for them in adapting the new practice fully (see Part 4 below).

Part 3: Kenneth and Lorenzo's Presentation of the Fish.com Task

In this section we discuss Kenneth and Lorenzo's presentation, which lasted for approximately 17.5 minutes and culminated in the reinvention of a bifurcation diagram. Our analysis of this reinvention details all five forms of activity that constitute the changing structure of the solution space practice. In particular, we show how use of dP/dt versus P graphs was at the core of this group's analysis, which stands in striking contrast to the analysis we saw Brady present in Part 2. In addition, we document how the emergence of the summarizing the changing structure of the solution space practice was made possible through the brokering actions of the teacher and of Kenneth and Lorenzo.

Episode 3a: Introducing a New Inscription

Lorenzo and Kenneth began their report with Lorenzo recapping their analysis of the differential equation before it was modified to reflect harvesting. They use three graphs in their explanation (Figure 7.6).

> Lorenzo: This here, that's, uh, that's our initial equation without any harvesting, without anything. For that equation [points to $dP/dt = 2P(1 - P/25)$ on their overhead presentation] we can see that if we graph, uh, dP/dt versus P, uh, that's going to look like, that's going to be a parabola with two equilibrium solutions. So we can graph that too [points to the P versus t graph], and that's where we start from.

The starting place for their report is similar to Kurt's presentation detailed in Part 1, but less detailed. In particular, Lorenzo used a single dP/dt versus P graph to ascertain the structure of the solution space, which is aspect "a" (Figure 7.1) of the summarizing the changing structure of the solution space practice. As we have previously argued, this is in and of itself a separate CMP that becomes embedded in this emerging new practice.

Moving on to the next part of their presentation slide, Lorenzo explained how they algebraically dealt with the parameter in the modified differential equation and then how they graphically treated this algebraic result.

> Lorenzo: Now, we include this negative k [points to $dP/dt = 2P(1 - P/25 - k)$ on their overhead presentation] which tells us that we are going to harvest some fish. That k can be any number so we're just going to leave that as a constant. Now, we can solve this equation. We can set dP/dt equal to zero, like we did in this case [when $k = 0$], and try to find the two equilibrium solutions, but in this case, since we've got a constant, there are going to be lots of equilibrium solutions. Actually, we can graph those. That's the graph that we got right here [points to graph of P versus k, see Figure 7.7]. So this here, that's, uh, the graph of equilibrium solutions versus k.

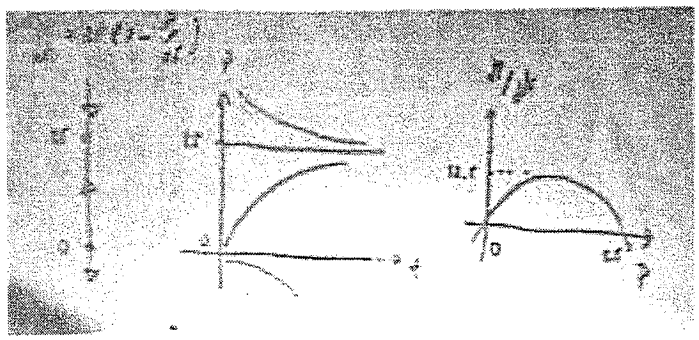


Figure 7.6. Analysis of initial differential equation.

In terms of the summarizing the changing structure of the solution space practice, Lorenzo just introduced to the class aspects "c" and "d" of the summarizing the changing structure of the solution space practice (Figure 7.1). This was the first time that these two specific ways of using the differential equation had appeared in class, and hence they represent Lorenzo and Kenneth's own creative efforts. Moreover, Lorenzo's introduction of these two forms of activity positioned him and Kenneth as brokers between their small group and the larger classroom community. As brokers, their presentation served the function of inserting new algebraic techniques and a new inscription, namely a "graph of equilibrium solutions versus k."

Lorenzo went on to explain how he and his group used the equations for P_1 and P_2 to interpret the number of equilibrium solutions as a function of the parameter k. This is aspect "c" of the emerging practice (Figure 7.1).

> Lorenzo: Okay, from these two equations here [referring to $P_1 = 12.5 + 2.5\sqrt{25 - 2k}$ and $P_2 = 12.5 + 2.5\sqrt{25 - 2k}$, as shown in Figure 7.7] we can see that if you want to have an equilibrium solution, this root here has to exist, meaning this number here [pointing to the discriminant in the P_1 and P_2 equations] has to be either positive or zero. That means that our k cannot exceed 12.5. So if you want to have an equilibrium, you can't exceed 12.5, you can't take out more than 12.5 fishes per cycle: a year, a month, or whatever it is. Because if you don't have an equilibrium, that means basically all your population is going to drop to zero [moves his hand straight down in a way that seems to resemble a flow line in case when $k > 12.5$ [see Figure 7.8]. Now, if you, if k is zero—

Here we see Lorenzo carefully explaining how to interpret the equations for P_1 and P_2 in terms of the equilibrium solutions. The detail that he provided stands in contrast to the very brief explanation given for the

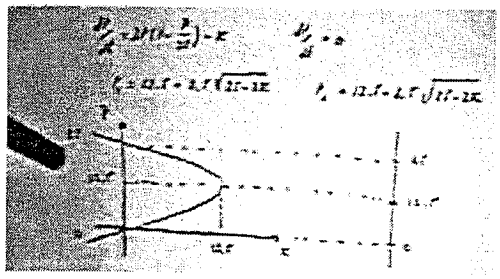


Figure 7.7. Initiating aspects c) and d) of the emerging new CMP.

initial situation when $k = 0$ and the graphs (Figure 7.6). This suggests that Lorenzo recognized the newness of his and Kenneth's approach and so he took some care in explaining their idea.

Episode 3b: Making Connections

Before Lorenzo could continue, the teacher interrupted him to inquire about the case when there are no equilibrium solutions.

> Teacher: Hold on, hold on. I just want to interrupt for a second. Lorenzo, you just said that if you don't have an equilibrium solution your population is going to drop to zero? Is that what [Lorenzo: Yeah]. So that was similar to what you guys were concluding before, right? If you're taking out fifteen or twenty. So if your k is bigger than 12.5 it's over here [points to a k-value bigger than 12.5 on the k axis of the P versus k graph]. How do I understand what—I mean, there's nothing here [waves hand over the region to the right of the parabola on the P versus k graph] on the graph that Lorenzo is showing, right? Is there, is there something I can still understand from that graph that he's putting up there?

We see this interruption as a significant brokering move that functioned to provide opportunities for students to reflect on new ideas and inscriptions. Part of reflecting on this new inscription of P versus k was to mark similarities across presentations. For example, the teacher noted that Lorenzo and Kenneth's conclusion when k is bigger than 12.5, "was similar to what you guys were concluding before." After noting this similarity, the teacher went on to prompt students to make sense of the P versus k graph when k is bigger than 12.5. In particular, the teacher pointed to a place where "nothing's there" in Lorenzo and Kenneth's P versus k graph and asked, "is there something I can still understand." The significance of this brokering move is evident next when a student responded to the teacher's query by seeking to make a connection

between different inscriptions, specifically between the P versus k graph when k is bigger than 12.5 and a dP/dt versus P graph.

> Dylan: What's that graph on the right there? Where the parabola's below P, down on the bottom right.
>
> Lorenzo: Yeah, that's uh, that's where I split this here [points to the dP/dt versus P graph on their overhead for the case when $k > 12.5$. See Figure 7.8].
>
> Dylan: Because that is when the [equilibrium] solutions are imaginary.

Dylan's exchange with Lorenzo was the start of a more detailed conversation about how to make connections across the various inscriptions. As we see next, Lorenzo took the lead in explaining the mathematical details regarding the connections between the P versus k graph, graphs of dP/dt versus P, and the algebraic equations for P_1 and P_2. As such, Lorenzo was acting as a broker between his small group and the rest of the class in order to articulate the underlying mathematical ideas. Lorenzo began his explanation by revisiting connections for the "starting case" when k equals zero.

> Lorenzo: Yeah, I started, I wanted to include all the possible combinations that we can have for k. For example, if k is zero, that means you don't take any fish out, and then this radical here [points to the equations for P_1 and P_2] becomes five, and this, then you get two equilibrium solutions, twenty-five and zero. That's our starting case—that means we don't harvest anything.

It is significant that Lorenzo chose to tell the class that he "wanted to include all the possible combinations" for k. Some of the previous group

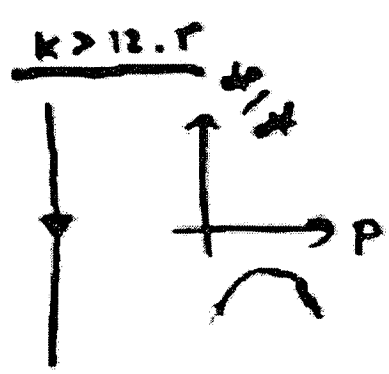


Figure 7.8. dP/dt versus P graph when $k > 12.5$.

presentations did not consider all possible cases for k. Thus, Lorenzo was providing a more thorough analysis than some of the previous presentations, and hence he may have found this worth noting. In addition, what constituted a new case for Lorenzo and Kenneth was when there is a change in the number of equilibrium solutions. That is, when there is a change in the structure of the solution space. Their P versus k graph offered the class a single graphical inscription that captured this change.

Next, Lorenzo explained some of the connections between the P versus k graph, the flow line, and the dP/dt versus P graph when $k = 12.5$.

> Lorenzo: If you take k to be exactly 12.5, which is this here [points to the vertex of the parabola in the graph of P versus k], then these two graphs of equilibrium solutions meet at this point here [see Figure 7.9a] and you have only one equilibrium solution [points to the place on the flow line when $k = 12.5$, see Figure 7.9(b)]. You can also see how if you graph dP/dt, that, uh, your population is going to be decreasing all the time except at one point [points to the vertex of the dP/dt versus P graph when $k = 12.5$, see Figure 7.9(c)]. That means basically the most fish you can take out is 12.5. That's the best case for you. But in case that your [initial] population is bigger than 12.5, because in that case you are going to come here and stop there [traces the flow line to the left of the dP/dt versus P graph shown in Figure 7.9b down to 12.5] and you are going to be taking 12.5 every time but your population still stays the same—12.5. If k is bigger than 12.5—

In the previous excerpt Lorenzo coordinated, in the case of $k = 12.5$, how to locate the one equilibrium solution on the P versus k graph, on the flow line, and on the graph of dP/dt versus P. He also interpreted this one equilibrium in terms of the fish.com scenario when he stated, "That means basically the most fish you can take out is 12.5." He then continued to explain that if the initial population was bigger than 12.5, then eventually the fish population would stabilize at 12.5. This explanation was accompanied by his tracing over the flow line, starting at a P-value bigger than 12.5 and ending at 12.5. As we see next, the teacher used Lorenzo's explanation of the flow line as a springboard to elaborate their P versus k graph.

Episode 3c: Teacher Initiates Aspect (e) of the Emerging New Practice

In the previous section we see how the teacher's brokering actions between Lorenzo's small group and the larger classroom community served the purpose of providing opportunities for the class to reflect on the new ideas and inscriptions that Lorenzo and Kenneth introduced. He did not, however, elaborate on what might be thought of as the mathematical content. It was Lorenzo who brokered the mathematical content by explicitly making connections between various inscriptions and equations. Lorenzo and Kenneth, however talented, were only able to go so far

in their role as brokers of the mathematical content. It is the teacher who was positioned to help further develop the mathematical ideas initiated by Lorenzo and Kenneth in ways that were increasingly compatible with more conventional or formal approaches. In this way, the teacher acted as a broker between the classroom community and the broader mathematical community. In this particular case, the teacher stopped Lorenzo from continuing to explain how to see the various connections when k is bigger than 12.5 and began to elaborate Kenneth and Lorenzo's P versus k graph by layering a flow line on top of their graph.

> Teacher: Hold on a second, I want to make sure that I understand. I was just thinking that—can I draw on this, is that okay? [Lorenzo: Yeah.] So, um, you were saying that if k was equal to 12.5—this is your flow line with one equilibrium solution [points to the flow line for $k = 12.5$ shown in Figure 7.9(b)] [Lorenzo: Yeah.]. And your population is, you start with bigger than 12.5, it would be decreasing to 12.5. Any population below 12.5, if you're harvesting 12.5, would—you'd die out, right? [Kenneth: yeah, so you'd have to start] So it seems to me like right here [points to the vertex of the parabola on the graph of P versus k], I could get that information in a way. So when k is twelve point five, we have one equilibrium solution [draws a dot at the vertex of the parabola on the k versus k graph], and then the population, your P, are all decreasing here [draws a line segment with downward-pointing arrow above the dot just drawn]. [Lorenzo: Yeah.] And then over here it's all decreasing as well, right? [draws a line segment with down-ward pointing arrow below the dot]

In the previous excerpt the teacher took the flow line associated with $k = 12.5$ and he layered it on top of the P versus k graph. In terms of the summarizing the changing structure of the solution space practice, we reference this form of activity as aspect "e" practice (Figure 7.1). We see the teacher's layering of the flow line on top of the P versus k graph as a brokering move between the broader mathematical community and the classroom community because it was the teacher who knew that bifurcation diagrams contain more information than the graph that Lorenzo and

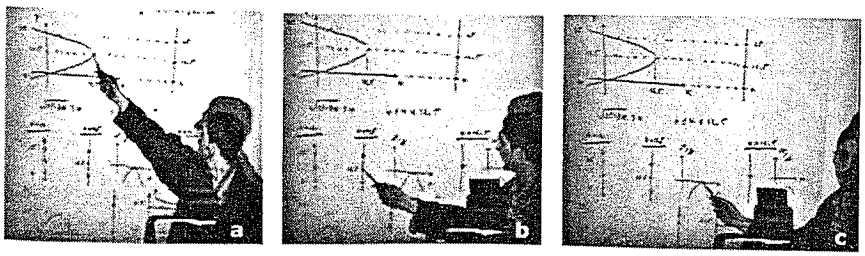


Figure 7.9. Making connections to the graph of P versus k when $k = 12.5$.

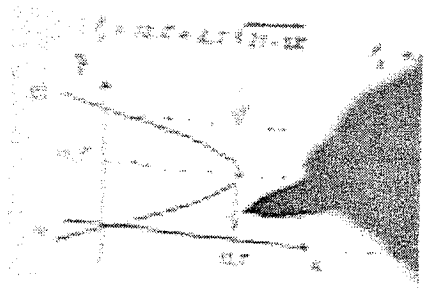


Figure 7.10. Layering a flow line on the P versus K graph.

Kenneth created—bifurcation diagrams also contain information on the equilibrium solution type. By layering the flow line for $k = 12.5$ on the P versus k graph, the teacher was adding important mathematical information about the structure of the solution space when $k = 12.5$.

For Lorenzo, the teacher's actions were apparently not a major revelation because he immediately commented, "Sure, that's exactly what I've got here," pointing back to the flow line (Figure 7.9b). The teacher continued his brokering actions by next asking how to understand the P versus k graph in a similar way when $k = 0$. Again, Lorenzo responded in a way that suggests that the teacher's layering of the flow line was not surprising to him:

> The same graph, that's, see, here, I just showed you a few particular cases, but from this graph [the graph of P versus k] we can basically see all those cases, because this is exactly the graph of equilibrium solutions.

The teacher responded to this by continuing his brokering role. However, here he switched back to brokering between the ideas put forth by Lorenzo's group and the classroom community, as opposed to adding layers of flow lines, which served the purpose of bringing the mathematics of the local classroom community more in line with that of the broader mathematical community.

> Teacher: Hold on a second. So Lorenzo just said from that graph we can see all particular cases. [Kenneth: Yeah] Help us out. Kristen? I'm sitting in class. I'm not sure I understand what he just said. Help me out.
>
> Kristen: I'm not sure I understand. How could you see, like, what if there was fifteen? Like he was saying earlier

Perhaps not surprising given the complexity and interconnectedness between the many inscriptions, Kristen was uncertain how one can "see"

all the cases as readily as Lorenzo apparently did. In the next Episode we detail how all the other cases can in fact be "seen" on the graph of P versus k.

Episode 3d: Teacher Hands Over Responsibility for Layering Flow Lines

Whereas it was the teacher who initiated the layering of flow lines on top of the P versus k graph, in this episode we detail how the teacher handed over responsibility of this task to Kenneth. While Kenneth had largely been quiet thus far in his group's presentation, here he stepped up and took over primary responsibility for discussing their report. To some extent, Kenneth's move to the center was prompted by the teacher, who, without Kenneth implicitly or explicitly requesting, gave the overhead marker to Kenneth and then stepped to the side of the room, putting Kenneth in charge. These subtle teacher actions fostered what Brousseau (1997) refers to as the devolution of responsibility. Rather than take on the question of how to interpret the case when $k > 12.5$, as suggested by Kristen, Kenneth chose to elaborate on the specific case of $k = 6$. He began his exposition by pointing to the equilibrium solutions on the P versus k graph corresponding to $k = 6$ (Figure 7.11): "You have a [harvesting rate] of six right here [points to where $k = 6$ on the P versus k graph], and you're going to have two equilibrium solutions right here and right here [draws in two dots on the P versus k graph corresponding to $k = 6$]." Kenneth's group had included graphs of dP/dt versus P, a flow line, and P versus t graphs this particular case at the bottom of their overhead presentation (Figure 7.12).

> Kenneth: So, like, down here, on the bottom [points to dP/dt versus P graph shown in Figure 7.12], you see that k equals six, our harvesting rate, right. So we have two equilibrium solutions. We graphed the differential equation against the population, so we get these two different equilibrium solutions here, right? [points to the two places that the graph of dP/dt versus P crosses the P axis]. So, if you were to graph P versus t for that, you'd see that you'd have an equilibrium solution here at 3.5 [points to the P versus t graph] and one here at 21.5. So you'd start out with a population, let's say a hundred, and then at that harvesting rate of six, you would just decrease to 21.5 over time. Similarly, if you were to start out with population anywhere between 3.5 or 21.5, you would just increase, or converge, to 21.5. So it's like, a sink, I guess you could say, would be our 21.5, and 3.5 would be our source. So if you were to start out anywhere underneath 3.5 for a beginning population with a harvesting rate of six, you would just decrease to zero over time.

Kenneth's explanation of the case when $k = 6$ address many of the features of what we referred to as aspect "a" of the emerging new practice. In particular, Kenneth details the nature of the two equilibrium solutions,

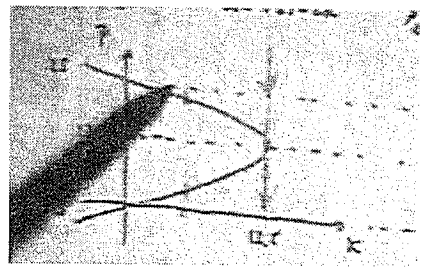


Figure 7.11. The P versus k graph when $k = 6$.

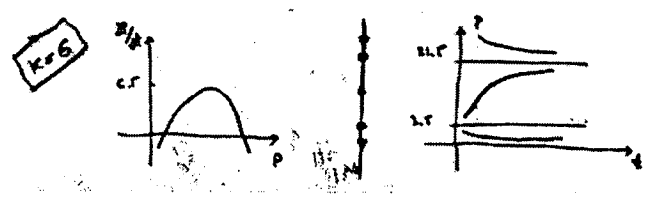


Figure 7.12. The case when $k = 6$.

explaining why 21.5 would be a sink and why 3.5 would be a source. Next, the teacher prompted Kenneth to show the class how to reinterpret what he just said on the P versus k graph. Kenneth understood this request to mean that he should layer a flow line on top of this graph, similar to what the teacher did for the case when $k = 12.5$.

Teacher: Alright, so show us this k equal to six up on this P versus k graph.

Kenneth: So. This is six right here [writes in 6 at the appropriate place on the k axis of the P versus k graph as shown on Figure 7.13]. So this population here is 3.5. And this one here, what did we say it was, 21.5 [writes in these values on the vertical axis of the P versus k graph as shown on Figure 7.13]

Teacher: So those would be my equilibrium solutions, you're saying, Kenneth?

Kenneth: Correct. So this graph [points to the P versus k graph] just shows you for any different harvesting rate in between zero and twelve point five the various, uh, population equilibrium solutions.

The usefulness of Kenneth's explanation was demonstrated by Jim, who spontaneously added how to interpret the space inside and outside the parabola on the P versus k graph: "So in a way, the, um, inner part of that parabola is where the population is increasing [Kenneth: Correct.] and outside is where the population is decreasing." Kenneth then followed up on Jim's observation by "dropping" a phase line on top of the P versus k

graph. Consistent with Jim's observation about the meaning of anywhere inside and outside the parabola, Kenneth then stated that you could drop a phase line "anywhere" on the graph, and chose to do so for $k = 6$. As such, the case when $k = 6$ represents a generic case for $0 < k < 12.5$.

Kenneth: Right. Yeah, if you were to drop, like, a phase line on top of the graph anywhere, like if we were to drop one here [draws a vertical line at $k = 6$, see Figure 7.14] you'd see that increasing in there [draws an upward facing arrow on the line just drawn inside the parabola], decreasing there [draws a downward facing arrow on the line above the parabola], decreasing there [draws downward facing arrow on the line below the parabola].

In Part 4 students revisit how to determine the direction of the arrows on the flow/phase lines that are layered or dropped on a graph of P versus k. From Kenneth and Lorenzo's perspective, the way one decides on the direction of the arrows comes from analyzing graphs of dP/dt versus P. This is a form of activity that Brady does not engage in without the brokering by Kenneth, Lorenzo, and the teacher. As Kenneth continued his explanation, he returned to Kristen's question about how to interpret the P versus k graph in terms of the fish harvesting scenario when $k = 15$.

Kenneth: So we can't harvest any more than twelve point five. You see, there's no population value associated with that [indicates the region to the right of the parabola]. You know, if you come out here to fifteen [writes in 15 on the k-axis and draws a vertical line at $k = 15$ without an arrow], there's no population out there, so,

Teacher: Could you drop a flow line, a phase line there?

Kenneth: You could drop them there, but they, I mean—

Teacher: Would there be a flow line or phase line? [Several students speak up at once. Someone says, "It's always decreasing," and someone else says, "It would just be all gone."]

Lorenzo: Of course, but it's just no equilibrium solutions.

As Kenneth, Lorenzo, and others in the class recognized, one could drop a phase line when $k = 15$, that is, at a k value where they are no equilibrium solutions. However, there seemed to be some reluctance to do so. This reluctance is understandable since it is the equilibrium solutions to which all other solutions converge or diverge. With no equilibrium solutions, there is less information to portray with a phase line. Importantly for the owners of fish.com, however, when $k = 15$ the solutions are "always decreasing" and hence the fish population is eventually "all gone," as students in the class pointed out.

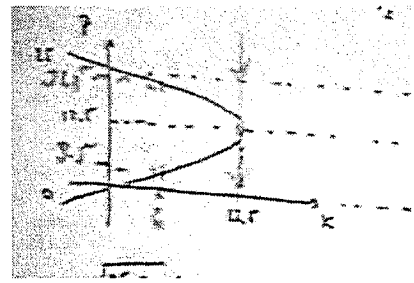


Figure 7.13. Marking equilibrium solutions on the P versus k graph when $k = 6$.

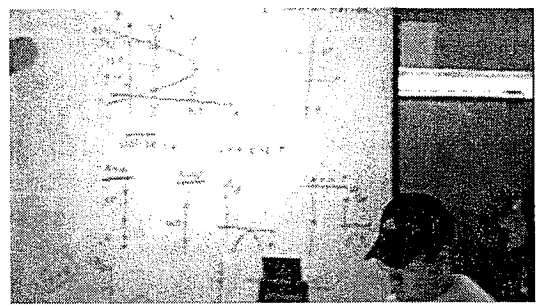


Figure 7.14. Kenneth "drops" a phase line on the P versus k graph.

Episode 3e: Reasoning with a Continuously Changing dP/dt Versus P Graph

After a brief discussion about what harvesting rate students would recommend to the owners of fish.com and why, Roy asked Kenneth and Lorenzo, "How do you know where to put those arrows?" Roy's analysis had included a slope field and phase line based on that slope field, but did not include a graph of dP/dt versus P. Kenneth interpreted Roy's question as a request to re-explain how to use a graph of dP/dt versus P to ascertain the direction of the arrows on a phase line, and proceeded to elaborate on his earlier explanation when $k = 6$. Lorenzo, on the other hand, interpreted Roy's question in a different way. While Kenneth interpreted Roy's question in terms of how to draw a phase line from a single graph of dP/dt versus P, Lorenzo followed up Kenneth's explanation by explaining why the phase lines are essentially the same whenever k is between 0 and 12.5. He did so without prompting from the teacher or anyone else.

Lorenzo: Now, maybe for better explanation. You see this first equation [points to the equation $dP/dt = 2P(1 - P/25) - k$] over there? And, uh, you graph dP/dt versus, uh, versus P, yeah. See that's, the first equation, is, uh, quadratic function. If you include a constant, in the quadratic function, you just shift,

Kenneth: It shifts it up and down [Moves his hand up and down in a continuous manner].

Lorenzo: You just shift it down. [Kenneth: Yeah.] So basically, all this here, as far as k is between zero and twelve point five. These graphs [points to the graphs of dP/dt versus P on their overhead presentation] are going to look the same. It's only these equilibrium solutions, their place, is going to change, like, you know, this [the gap between the equilibrium solutions] can shrink a little bit [makes a pinching gesture over a graph of P versus t to dynamically show how the equilibrium solutions come together as k changes].

Lorenzo began his explanation with, "Now, maybe for a better explanation." We do not know what made this a better explanation for Lorenzo, but in our view, Lorenzo's explanation was "better" because he explicitly attended to a continuously changing dP/dt versus P graph in relation to the structure of the solution space. Kenneth also participated in this explanation by animating the change to the dP/dt versus P graph by raising and lowering his hand. In terms of the summarizing the changing structure of the solution space practice, Lorenzo and Kenneth's elaboration is captured in activity "b" of this emerging new practice. Essentially, Lorenzo explained how the structure of the solution space would remain the same for $0 < k < 12.5$, with the only difference being that the gap between the equilibrium solutions would "shrink a little bit."

In the five preceding episodes of this Part 3 we see many brokering moves by Kenneth and Lorenzo in terms explicitly inserting new mathematical ideas into classroom discussion. In this episode we similarly argue that Kenneth and Lorenzo continued such brokering between their small group and the larger classroom community by explicitly reasoning with a dynamically changing dP/dt versus P graph in order to infer changes to the structure of the solution space. In the next and final episode of Part 3 we see the teacher build on the mathematical ideas inserted by Lorenzo and Kenneth.

Episode 3f: Introducing Conventional Terminology

In this last episode of Part 3, the teacher began by rephrasing what Lorenzo and Kenneth just said, and then connected this recap to formal or conventional terminology. In this way, we see the teacher again acting as a broker, first between Kenneth and Lorenzo's small group and the larger classroom community through his rephrasing and then between the entire classroom community and the broader mathematical community by the insertion of formal or conventional terminology.

Teacher: Let me just say something now. I think that was really nice. I want to pick up on something that Lorenzo just said [Teacher moves from the side of the room to the front]. He said that if you have different k values between zero and, well almost, less than 12.5, he's saying that the flow lines are all basically the same. [Lorenzo: Yeah.] Essentially there's two equilibrium solutions, one of which is a sink and one of which is a source. So no matter whether your k harvesting rate is between zero and 12.5, not including 12.5, you're going to have two equilibrium solutions, like here [points to the P versus t graph for $k = 6$], one of which is a sink and one of which is a source so that basic structure [holds out two flat hand to indicate the sink and source equilibrium solution as show in Figure 7.15a] and the whole way in which the population versus time changes is similar. These [the equilibrium solutions] move together, is what you're saying, right? [Lorenzo nods] And then at this particular value of 12.5 for the harvesting rate something happens. Those two equilibrium solutions—

Kenneth: Become one.

Teacher: Become one [brings his hands together as shown in Figure 7.15c].

In Lorenzo's explanation about the change to the equilibrium solutions as k changes, he used a somewhat subtle pinching gesture to animate the changing structure. The teacher, in contrast, made a much more obvious gesture as shown in Figure 7.15. Based on the teacher's placement of his hands away from his body and over the graphs on the overhead, we suspect that the teacher's gesture was intentional, meant to be clarifying and informative for students, as opposed to a gesture that is spontaneous and idiosyncratic. As the teacher continued, he introduced the "technical" word for the value of the parameter when there is a change to the structure of the solution space.

Teacher: There's a technical word for that parameter value—It's called a bifurcation value. So this k is a parameter. This k's a parameter and this value right here [circles 12.5], the technical word for that, that occurrence, when your two equilibrium solutions change to one, is a fundamental change in the way in which all the flow lines look. They were saying basically all the flow lines in here look exactly the same, their equilibrium solutions are just pushed together a little bit. But when k is 12.5, there's a

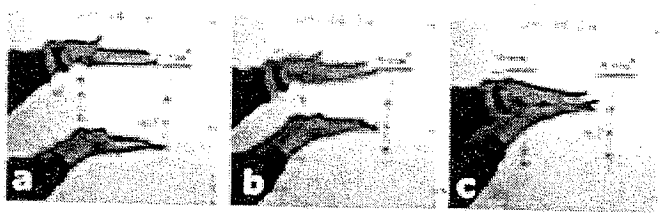


Figure 7.15. Animating the changing structure of the solution space.

> shift, or a change, in the way the structure of the flow line is. That parameter value is called a bifurcation value [Writes "bifurcation value" on the overhead]. And then anything above 12.5, the flow-line, well we didn't put it in here but from what you guys have said, the flow line here is [draws in the missing arrow on the vertical line previously drawn by Kenneth on the P versus k graph when $k > 12.5$], there are no equilibrium solutions, right? The population is just going to be dying out [points downward along the flow line just drawn]. Right. So there's a flow line here, it's just totally decreasing. [Lorenzo points to the flow line and dP/dt versus P graph for $k > 12.5$.] Exactly, right, there it is. Right, so, this, that parameter value here at 12.5, again, is called a bifurcation value, and it just sort of helps us get a sense about when there's a big change, so here,

The class has now produced what an expert would recognize as a bifurcation diagram. Lorenzo and Kenneth first introduced what we might now in retrospect call an "empty" bifurcation diagram, and through the brokering actions of the Kenneth, Lorenzo, and the teacher, this empty bifurcation diagram was elaborated on by dropping phase lines on top of the graph. This elaboration was initiated by the teacher and furthered by Kenneth, Lorenzo, and Jim. Finally, in contrast to more traditional teaching in which formal or conventional terminology is often the starting place for students' mathematical work, this teacher chose to introduce the formal mathematical language only after the underlying idea had essentially been reinvented by students.

In detailing the emergence of this bifurcation diagram, we articulate all five forms of mathematical activity that constitute the summarizing the changing structure of the solution space practice (Figure 7.1). In Part 4 we see how all five forms of activity are relived in a new problem, one not couched in a real world scenario, and how Brandon and his group become more central participants in the new practice.

Part 4: Brady's Group Becomes More Central Participants in the New Practice

In this section, we detail Brady and Neil's presentation of their analysis of a new differential equation, $dy/dt = ky + y^3$. Brady and Neil incorporated important aspects of the previous solution of Kenneth and Lorenzo, but also failed to use the important tool of the dy/dt versus y graph to interpret the structure of the solution space. In the class discussion the teacher, and at times Lorenzo and other students, pushed Brady and Neil to use this tool to correct aspects of their solution and further interpret their results. In Episode 4a, we describe the aspects of the summarizing the changing structure of the solution space practice that Brady and Neil used in their presentation and comment on which of these were new practices for their

group compared to their previous presentation. In Episodes 4b–4d, we describe how the new CMP continued to be disseminated to Brady and Neil and the classroom community through the brokering of Lorenzo, other students, and the teacher.

Episode 4a: Brady and Neil's Presentation

Brady explained what he and Neil had written on their overhead slide and answered clarifying questions from the teacher and other students. In this way we see Brady and Neil illustrating what aspects of the CMP that their group worked with and which they did not work with. Brady began with setting $dy/dt = ky + y^3$ equal to zero and solving to find $y = 0, y = \pm\sqrt{-k}$: "It made sense that when dy/dt is equal to zero you have an equilibrium point. So all we did was we set dy/dt equal to zero, and solve the equation to find out what y is equal to when dy/dt is zero. We got y equal to zero and y is equal to plus or minus square root of negative k."

In Part 2, Brady's group had presented a table with "POP MIN" and "POP MAX" values and had noted that these were equilibrium solutions. So it is likely that when Kenneth and Lorenzo initiated what we describe as aspect "c" of the new CMP (Figure 7.1) that this was a sensible strategy for Brady's group. Having implemented a similar strategy for specific values of the parameter on their own, it made sense to Brady's group to generalize this to an equation with the parameter as a variable as Lorenzo and Kenneth had done. Brady continued his explanation by describing how the graph on the overhead of y versus k illustrated these equilibrium values.

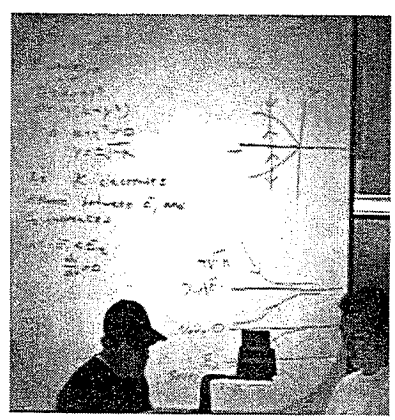


Figure 7.16. Brady and Neil present their work on $dy/dt = ky + y^3$.

Brady: Then we did the graph of all three of those functions and we got this here [points to y versus k graph with phase line drawn in]. And what we found was that as k got negative ... it decreased, then the gap between the two equilibrium points and the center equilibrium point became wider and wider. So we could make the statement that says, As k decreases the distance between E_1 and E_2 increases [Figure 7.17].

Here Brady illustrates aspect "d" (Figure 7.1) by describing how they have plotted the equilibrium values as they vary as a function of the parameter. This is not something that their group had done in Part 2. From the two previous excerpts we see that Brady and Neil had engaged in aspects "c" and "d" (Figure 7.1) in ways that they had not done earlier in the class period for the fish.com problem. However, there was no evidence on their overhead or in their discussion about using a dy/dt versus y graph, either with or without a parameter (aspects "b" and "a"). In Part 2 there was no evidence of Brady's group taking part in aspects "a" and "b" of the practice in their solution to the fish.com task. This lack of reasoning with dy/dt versus y graphs served as a constraint for them in adapting the new practice fully, as we will see in Episode 4b. The extent to which they have implemented aspect "e" is discussed in Episode 4b.

Episode 4b: Lorenzo and Others' Brokering Moves Regarding the Flow Lines

Lorenzo questioned the direction of the arrows that Brady and Neil drew on the graph of y versus k. Brady had not yet described these in his

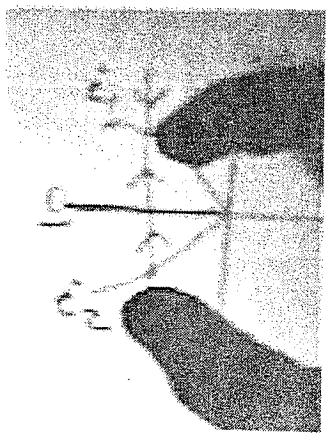


Figure 7.17. Brady illustrates the distance between E_1 and E_2 increases as k decreases

presentation of the overhead, but the overhead included an attempt to put arrows on the y versus k graph in a way that mimicked Kenneth and Lorenzo's use of flow lines. Brady and Neil's arrows pointed up between the E_1 and E_2 curves and down when y was larger than E_1 or smaller than E_2 (Figure 7.17). This matched the direction of the arrows in the problem that Lorenzo and Kenneth illustrated in that they were up inside the curve and down outside of it (Figure 7.14).

Teacher: Lorenzo has a question for you.

Lorenzo: I think you guys got a problem with uh ... (laughing in the room)

Teacher: Why do you all laugh? Don't laugh.

Brady: What's your, what's wrong with it?

Lorenzo: With the direction of those arrows on the flow line.

Brady: What's wrong with them?

Lorenzo: Well, you have two wrong directions. Okay, you've got four arrows. [Brady: Right.] How did you figure it out?

Brady: Oh, what is it ... you're talking about these right here? [points to the y versus t graphs]

Kenneth: No. The, the phase line ...

Jim: The way your arrows are going through your flow line.

Lorenzo: Two of them are right. Two of them are wrong.

Brady: Which two are right, you think?

Lorenzo: And uh, if you go from the bottom, that one is right. The next one (Brady points to the phase line) is right then. The other two are, yeah, should be the opposite way.

Teacher: How can we decide—

Kenneth: Yeah, how, how do we know—

Teacher: How can we decide?

From the above we see that Brady was not aware of the error and did not immediately have a way to think about this issue. Below we see that Brady responded by pointing to the y versus t graphs. The direction of those graphs did match the arrow directions on the y versus k graph. He also reverted back to a numeric focus for specific values of the parameter reminiscent of the numeric focus of his group in Part 2: "You can look at the function [points to the y versus t graphs].... No, I—it would, well—yeah, because it would increase because if you had put in negative nine you would get three. If you put in negative four, you'd get two. So it [points to the top curve of the graph of the equilibrium solutions] goes up." Here we see brokering regarding aspect "e" (Figure 7.1). Brady's group had imitated this aspect of the work done by Lorenzo and Kenneth's group. Also, they seemed to know that these arrows were representative of the direction the

y versus t functions moved, at least in some sense. However, they were not re-presenting the true structure of the solution space for the equation $dy/dt = ky + y^3$.

In this discussion, the line on the overhead graph of y versus k and its arrows served as a *boundary object* between the classroom community as a whole and the small group of Lorenzo, Kenneth and the teacher. Each group interpreted the boundary object differently. The small group, and perhaps some other students in the class, had come to understand a way to interpret this object with reference to the parameterized equation and the y versus k graph. Other students in the class, at least Brady, seemed to have a way to interpret the line as related to the y versus t graph but not with regard to the parameterized situation. Lorenzo and others pointing to this boundary object and questioning about the arrows served as acts of brokering between two communities about aspect "e" of the new CMP.

Episode 4c: Teacher and Others' Broker Moves Regarding the Use and Interpretation of a dy/dt versus y Graph

This episode began, immediately following Brady's comment above, with the teacher explicitly asking Brady and Neil to graph dy/dt versus y and interpret it: "Can you give us a graph of dy/dt versus y for k is equal to negative nine? That might help us decide whether or not the rate of change is positive or negative." After a couple moments of stumbling, Brady rewrote the equation for $k = -1$ and drew a correct dy/dt versus y graph for it. With the dy/dt versus y graph drawn, Lorenzo prompted, "Do you see what I'm saying now?" Brady seemed to immediately know how to interpret the graph of dy/dt versus y to say more about the flow lines and the solution space.

Brady: Well, it's uh, it's increasing here [draws a plus sign over $-1 < y < 0$] and it's increasing there, right? [draws a plus sign over $y > 1$]. Negative there and negative there. So really, maybe we did mess up the top. So it really should go like that [switches the arrows at the top of the phase line on the y versus t graphs (see Figure 7.18)] and like that—

Lorenzo: That's it.

Brady: And then that, like that. So really then that messes up all this [points to the y versus t graphs].

Kenneth: [softly] Yeah.

In this way Brady showed that he was able to competently engage in aspect "a" of the new CMP. This is not surprising in that aspect "a" was already itself an established practice in the class. However, this seems to

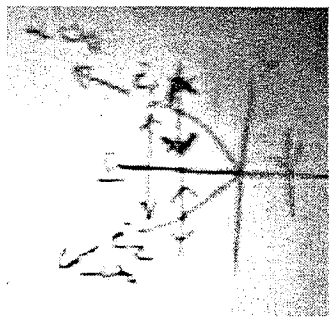


Figure 7.18. Brady's corrected flow line.

be the first time that Brady had engaged in this practice in the context of discussing a parameterized situation. That is, we see here how the teacher calls for the use of the *dy/dt* versus *y* graph in a way that brokered between Brady and Neil's reasoning and that of Lorenzo and Kenneth. Here the *dy/dt* versus *y* graph served as a boundary object in that it was a familiar object for both groups, but one that had been interpreted and used by Lorenzo and Kenneth in their presentation in a way that Brady and Neil had not yet done. The teacher's act of calling on Brady and Neil to interpret a particular *dy/dt* versus *y* graph in this setting began the process of bringing Brady and Neil into more central participation in the new CMP by having them engage in part a) of the new CMP in this context.

Episode 4d. Teacher Broker Moves Regarding Interpreting Changing dy/dt Versus y Graphs

In this section, aspects "b," "d," and "e" of the new CMP begin to become more established in the class, including for Brady who, as the person at the overhead, functioned as representative of the students in the class who had not yet become a central participant in the new practice. Initially the teacher asked fairly direct questions to bring out a clarification of several important ideas and relationships involved in interpreting the *y* versus *k* graph.

Teacher: So how many, when *k* is negative, how many equilibrium solutions do you have?

Brady: When *k* is ne—we have three.

Teacher: Three. Does it matter whether *k* is negative ten or negative four or negative.... [Brady shakes his head says "Nope"] you've got three equilibrium solutions.

Teacher: What are they: sinks, sources, nodes?

Brady: The bottom one's a source. The second one is a uh ... sink. And the third one is a source.

Teacher: So I want to relate that back to something that Kenneth said, is that no matter what your negative k value was here. No matter what your k value is here, if it's negative, you get the same basic flow line. Just the distance be, distance between those equilibrium solutions is the same. Now that, in the previous problem, the bifurcation value, that special value of the harvesting rate, that parameter value was twelve point five. What's the, what's the special value of the parameter here, where there's a change in the kind of flow line you get?

Neil & Brady: Zero.

The above excerpt did not explicitly refer to the dy/dt versus y graphs, but it seems that Brady, having corrected the arrows using the dy/dt versus y graph for $k = -1$, then had a sense of what that means for all negative values of k. Thus the teacher's questions served to help Brady, and perhaps other students who were thinking like Brady, to move toward more central participation in aspect "b." Immediately following this excerpt the teacher engaged in another type of brokering by trying to engage other students in the class in the discussion. The teacher tried to get a student named Ted to take a more active role in the conversation, and then the teacher worked to get other members of the class to explain more about what a bifurcation value is and what it indicates about a solution space.

Teacher: Who else wants to do me a favor, just summarize the question that I asked. Dylan, yeah. Go ahead.

Dylan: Where is the bifurcation point?

Teacher: That's, that's clear and crisp, but say a little bit more about ...

Dylan: Where does the nature of your flow lines change? 'Cause on the left, you see there every flow line, every vertical line you take, um, is the same. Where does that change occur? Where the actual type of lines that you have will be different with the arrows.

Teacher: That's, that's a really nice way to say ... to somewhat elaborate on what does it mean to say what is that special value, that parameter, that bifurcation value where you have a change in the kind of flow line? [Pause] And so I heard, in the back here they said, "k is equal to zero." [Brady circles the point at which k is equal to zero]. Do you agree with that? Do you disagree with that as a, as the bifurcation value?

Brady: That's the point where all three graphs meet. Or where the three equilibrium solutions meet.

The teacher worked to help students better understand the formal term "bifurcation value." In this way he brokered between the mathematics community and the classroom community. The teacher and other students also continued to broker between those more central to the new practice

and those more peripheral participants. To encourage a further elaboration of the nature of the changing space of solution functions, the teacher asked the students to explain more about the nature of the solution space when k is positive. This discussion began with further interpreting the equilibrium values in terms of the y versus k graph, aspect "d" of the CMP (Figure 7.1).

Teacher: And what happens as k gets a little bit bigger, then?

Brady: As k gets positive? [Teacher: Yeah.] Then you just have uh, dy/dt is uh—[Kenneth: Negative for all time.]—is negative.

Teacher: So do you have any equilibrium solutions after that?

Brady: No.

Teacher: Kurt, say, say it out loud to the class.

Kurt: Don't you have one still at y equals zero?

Vincent: Yeah, at y equals zero we have one.

Brady: Yeah, but I mean like as, as positive.

Kurt: At positive values of k, you're still going to have y equals zero.

Brady: Oh, that's right. [Lorenzo: Uh-huh.] I didn't even think about that.

Brady initially thought that the situation for positive k values would be the same as in the fish.com problem in which there was no equilibrium value and dP/dt was negative for all values of P. Here, with the help of other students, Brady realized that $y = 0$ is still an equilibrium value when k is positive. As the conversation continued, the teacher requested the class to think about the flow lines. Other students brokered the notion that the dy/dt versus y equations and graphs can still be used to determine the direction of the arrows. Although using dy/dt versus y graphs had been discussed before, Brady had not yet interpreted the dy/dt versus y graphs for the situation where k is positive and had difficulty immediately seeing how to do so. After four suggestions and comments from the teacher and eight comments from four different students, Kenneth described in more detail how to use the dy/dt versus y graph to interpret the flow lines for positive k values.

Kenneth: We can, we can plug values into the, the dy/dt. Like plug zero in there. So we have just dy/dt equals y cubed. Which is uh similar to the graph on the bottom left. But there's no, um, little humps. It just uh, just kind of like, I mean you know what a y cubed looks like? So, we have increasing, when, uh ... when y is positive and decreasing when y is negative.

After Kenneth's comment, the teacher moved to the board and made a few clarifying points to broker between the notions proposed by Kenneth and other students who were trying to explain about using the dy/dt versus

y graph and those students, like Brady, who had not yet worked through interpreting the dy/dt versus y graph for positive values of k. The class ended with the teacher summarizing the nature of the equilibrium solutions and flow lines as k moved from being negative to zero to positive. In the next section, we distill the kinds of brokering moves that functioned as a mechanism for the initiation and emergence of the summarizing of the changing the structure of the solution space practice.

CATEGORIES OF BROKERING

In this chapter, we detail the emergence of the summarizing the structure of the solution space practice and the role of brokers in its initiation and ongoing evolution. This practice culminated in the collective creation of a complex and sophisticated inscription, namely that of a bifurcation diagram. Central to the emergence of this practice was brokering, which we demonstrated was a mechanism underlying the joint production of meaning. In this section, we step back from the fine-grained description of the emergence of the summarizing the structure of the solution space practice and reflect on what we see as more general brokering categories.

By definition, a broker is someone who has membership in more than one community, and brokering occurs when this person facilitates the infusion or appropriation of some form of activity from one community into another. The three communities that we identify are that of the broader mathematics community, the local classroom community, and the various small groups that comprise the local classroom community. The teacher in our data had the unique status of being the only person who was either a full or legitimate peripheral member of all three communities. Critical brokering moves, however, were not limited to the teacher. The undergraduate students in this class, who were full members of the local classroom community and their respective small groups, also carried out several noteworthy brokering moves.

Reflection on the various broker moves led to identification of the following three broad categories of broker moves: creating a boundary encounter, bringing participants to the periphery, and interpreting between communities. We follow Wenger (1998) in distinguishing between the terms boundary and periphery. The term boundary is more closely aligned with possible *discontinuities* between communities, whereas the term periphery is more closely aligned with possible *continuities* between communities. We hasten to point out that discontinuities and continuities are not dichotomies, but rather two ends of a continuum. Moreover, actual experiences of encounters between communities are simultaneously filled with a multitude of continuities and discontinuities.

We refer to the first general brokering move category as "creating a boundary encounter." A boundary encounter refers to direct encounters such as meetings, conversations, or visits between communities. Any boundary encounter will involve boundary objects. Boundary objects refer to objects that serve as an interface between different communities. A broker, by virtue of his or her membership in more than one community, is in a position to bring forth boundary objects that can facilitate encounters between communities. Wenger's examples of boundary encounters all entail direct meetings, conversations, or visits between communities. We adapt this notion to also include *indirect* encounters between communities. For example, except in rare cases, it is not feasible for direct encounters between the broader mathematics community and classroom community to occur. Instead, the teacher as broker can offer *indirect* opportunities for the classroom community to encounter the broader mathematical community. The teacher can do this by offering opportunities for students to engage in the broader discipline practices of modeling, symbolizing, defining, algorithmatizing, or proving. Such encounters allow for the possibility of participation in the authentic practice of mathematics, and hence provide occasions for the local classroom community to indirectly encounter the broader mathematics community.

We highlight two exemplary instances of this category of broker move from our analysis. The first example is between the local classroom community and broader mathematics community while the second example is between the classroom community and Lorenzo and Kenneth's small group. Our first example of creating a boundary encounter focuses on the teacher as broker in his role of selecting and constituting tasks. Because the teacher is a member of both the broader mathematics community and local classroom community, he is in a position to recognize characteristics of tasks that are likely to be productive for enabling newcomers to become more central participants in the discipline practices of mathematics, such as modeling and symbolizing. For example, the fish.com problem was recognized by the teacher as a task he could use to engage students in modeling by giving them an opportunity to revise the original differential equation to fit new assumptions. Modifying a differential to fit new assumptions was a novel task for students. In this case, the set of new assumptions and the original differential equation functioned as a boundary object because it allowed the classroom community to interface with the discipline practice of modeling.

After settling on a modified differential equation, the way in which the teacher constituted the remaining portion of the task was crucial in actually creating a further boundary encounter. In particular, the teacher constituted the task in such a way that it opened up the possibility for students to engage in symbolizing. Specifically, the teacher invited students

to develop a report for the new owners of fish.com, rather than requesting students to recommend a harvesting rate. The later request would more likely lead to discussions about specific numerical values, rather than creating their own inscriptions to tell the story of what happens for all possible harvesting rate values. Constituting the task as a report to the new owners, without prescribing what form this report was to take, allowed students to develop their own inscriptions. Thus, the modified differential equation and task to develop a report functioned as a boundary object because it provided the classroom community an opportunity to encounter the mathematics community via participating in the discipline practice of symbolizing.

The second example of creating a boundary encounter comes from our analysis of Lorenzo and Kenneth's presentation of their work on the fish.com task. In inquiry-oriented classroom settings, it is fairly common for particular small groups to present their work on a problem to the entire classroom community. Such presentations represent the opportunity for a boundary encounter between one small group and their local classroom community. In our experience, not all such small group presentations realize this opportunity. For example, a small group presentation that simply reports back to the class what their group did without a substantive exchange of ideas and interpretations leaves the interface between communities somewhat empty. In order to fulfill the potential for a boundary encounter, there has to be some boundary object that leverages differences between the communities and actions by brokers to encourage an exchange of ideas and interpretations between communities. The P versus k graph presented by Lorenzo and Kenneth's small group was such a boundary object because this particular inscription was entirely new to the rest of the class and was the center of a substantive exchange between Lorenzo's group and the whole class.

In Episode 3a we highlight the detail and care in which Lorenzo explained his group's novel graph of P versus k. Our analysis describes how this presentation introduced two forms of activity that comprised the summarizing of the changing structure of the solution space practice, and how the introduction of these two forms of activity positioned him and Kenneth as brokers between their small group and the larger classroom community. Lorenzo and Kenneth's small group also included the teacher as a legitimate peripheral member. This is noteworthy because while Lorenzo's careful explanation of their P versus k graph opened the door for the classroom community to interface with their new inscription, it was the teacher who pushed the door open even further. As we detail in Episode 3b, the teacher interrupted Lorenzo and Kenneth's presentation to make explicit a connection to a previous small group presentation and to raise questions about how to interpret the novel inscription introduced by Lorenzo and Kenneth.

This move functioned to provide opportunities for students to reflect on the new forms of activity presented by Lorenzo and Kenneth. Thus, the first part of Lorenzo and Kenneth's presentation described in Episode 3a, together with the teacher's question about how to interpret their P versus k graph, created the opportunity for a boundary encounter between Lorenzo and Kenneth's small group and the classroom community. As our two examples of the more general brokering category of creating a boundary encounter illustrate, this particular type of brokering move sets up the opportunity and conditions for the boundary encounter to be realized. Our remaining two categories of brokering moves exemplify different ways that encounters between communities are realized.

The second general brokering move category we identified is "bringing participants to the periphery." Broker moves in this category help or encourage participants to move toward another community along a continuum. This is in contrast to the first category in which participants were set up to encounter the other community in a way some difference or discontinuity. In the preceding examples the broker's role was to create the opportunity for participants to engage a boundary object (task, inscription) that was relatively new to them. This set up a boundary encounter in the sense of being an encounter between discontinuous, distinct communities—those who would already know how to engage the task versus those who would not or those who had created and interpreted an inscription versus those who had never seen it before.

In comparison, "bringing participants to the periphery" is about moving along a continuum between communities. We tender two examples of this category of brokering move. Our first example of bringing participants to the periphery highlights the interplay between the local classroom community and the mathematical community. In Episodes 3c and 3d we see the teacher layering a flow line on top of the P versus k graph created by Lorenzo and Kenneth's group and then handing over responsibility for layering the flow lines to Kenneth. The initial layering of a flow line was a brokering move by the teacher that served more as the creation of a boundary encounter. This initial step introduced a new interpretation to the P versus k graph that brought it closer to the mathematical community's notion of a bifurcation diagram. The initial introduction of something completely new in this way is a brokering move that presents a discontinuity. However, as the teacher encouraged Kenneth to make further interpretations of phase lines for the P versus k graph this became more of a process of encouraging Kenneth, and the class with Kenneth as their representative, to reason and symbolize in ways that were more consistent with the broader mathematics community.

For our second example of bringing participants to the periphery highlights brokering moves between the local classroom community and

smaller groups within this community. Parts 1 through 4 all include examples of the teacher acting as a broker to encourage members of one community to engage ideas of another community or for members of one community to explain their ideas more fully to another community. In this way the teacher is requesting that participants move toward another community through a continuous periphery. For example, in Episode 2b, after Brady has completed his initial explanation, the teacher said, "Brady, let's pretend that I'm an owner and I make a lot of money, and I'm an executive, but I'm not so sophisticated with tables and stuff. So I need some help understanding that table that you've got up there." This request for clarification asks Brady to extend the ideas of his group in more detail and more clearly to the rest of the classroom community. A few minutes later after further explanation from Brady, the teacher asks the class, "So that's a question to you all—does that explain what his table is? If you're the owners, are you understanding what he's, what information he's provided?" In this way the teacher requests the members of the classroom community to engage in the ideas of Brady's group and thus bring themselves toward the periphery between Brady's group and the classroom community.

The third general brokering move category that we identified is "interpreting between communities." As the label for this category implies, this particular category of broker move is one in which brokers facilitate the understanding of one community regarding how ideas are construed, notated, related, or labeled by another community. In comparison to the first brokering category, creating boundary encounters, this third type of brokering move occurs when a broker takes specific steps to fulfill or realize the opportunities that the creating boundary encounter moves offered. We provide two central, illustrative examples of this particular brokering category from our analysis. The first example involves brokering between the local classroom community and Lorenzo and Kenneth's small group. In the second example the brokering takes place between the local classroom community and the broader mathematical community.

Our first example of interpreting between communities comes from our analysis in Part 3. In Episode 3b, we see Lorenzo respond to Dylan's question about a particular dP/dt versus P graph (where $k > 12.5$). Lorenzo's response to Dylan's question went well beyond an answer to the particular question. As our analysis detailed, Lorenzo, acting as a representative for his small group, elaborated how the case where $k > 12.5$ fit with the other cases and how their group understood various connections to their novel P versus k graph. This explanation was significant because it served the purpose of framing how others' analyses could be interpreted in terms of one or more of these cases.

Lorenzo continued connecting his group's work to that of the others by explicating various relationships for the case when $k = 12.5$. Specifically, Lorenzo pointed to the vertex of the P versus k graph (Figure 7.9a) and explained that, "If you take k to be exactly 12.5 ... these two graphs of equilibrium solutions meet at this point here." Here Lorenzo made an explicit connection to their algebraic equations for $dP/dt = 0$ and their novel graph of P versus k. Lorenzo continued his explanation by purposefully linking the P versus k graph to two other inscriptions with which the class was more familiar. As he pointed to the node on the flow line (Figure 7.9b), he stated, "and you have only one equilibrium solution." He immediately connected this to a dP/dt versus P graph when he stated, "You can also see how if you graph dP/dt, that, uh, your population is going to be decreasing all the time except at one point," as he pointed to the vertex of the parabola (Figure 7.9c). Lorenzo was careful in his pointing, and through this care, he linked the inscriptions together by explaining how to construe the same information from three different graphical inscriptions. We see Lorenzo's pointing as a type of linking gesture that facilitated his efforts to interpret between communities. Linking gestures are often used to "provide conceptual correspondences between familiar and unfamiliar entities" (Nathan, 2008, p. 376). Here we see that Lorenzo was able to provide conceptual correspondences from what his classmates were already familiar with to a new, unfamiliar inscription through a careful use of pointing gestures. In relating what was familiar to the class to what was unfamiliar, Lorenzo's use of linking gestures facilitated his efforts to interpret between communities.

Our second example of interpreting between communities highlights the teacher's unique brokering role as the only person who is a member of both the broader mathematical community and the classroom community. Given this unique position, it is the teacher who can (re)interpret the mathematical ideas that are emerging in the local classroom community in terms of the conventional or formal terminology used by the broader mathematical community. In this way, the teacher can infuse formal terminology into the discourse of the classroom community.

In Episode 3f, we see the teacher setting up the infusion of the term bifurcation by using linking gestures to connect the familiar with the unfamiliar. One of the most familiar inscriptions for the classroom community was that of the P versus t graphs. The term bifurcation was unfamiliar to the classroom community, however the fact that the structure of the solution space is different for different k values was becoming increasingly familiar for students. In Figure 7.15, we see the teacher use a series of gestures that link the changing number of equilibrium solutions to the term bifurcation. In particular, the teacher extended his hands and forearms in a parallel manner to portray the parallel equilibrium solutions on the P

versus k graph, and then he brought his hands and forearms together (Figure 7.15c), at which point he explained that the "technical" term for the parameter value at which there is a change in the number of equilibrium solutions is "bifurcation value." Through these moves, the teacher explicitly introduced the conventional or formal term "bifurcation" at a point in the classroom discussion when it served the function of labeling an idea that was an emerging part of students' mathematical reality. We see this as a noteworthy departure from teaching that often starts lessons with formal or conventional terminology because it enables students to see themselves as capable of participating in the cultural practice of mathematics.

In this section, we introduce three generalized broker move categories: creating boundary encounters, bringing participants to the periphery, and interpreting between communities. Each of these brokering categories highlight the view that teaching and learning mathematics is a cultural practice, one that is mediated by and coordinated with the broader mathematics community, the local classroom community, and the small groups that comprise the classroom community. Because these categories were developed out of two days of classroom data, we make no claim that these categories are exhaustive. Furthermore, we contend that both the course content and the timing of the data observed influenced the categories' formulation. Thus, we expect that observing other data sets would result in the creation of additional broad categories or in the facilitation of a sharper definition of the existing categories through the creation of subcategories. It is to this end—observing more data sets for the expansion of the categories as well as for a sharpening of the existing categories—that we anticipate a direction for future research.

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NOTE

1. An autonomous differential equation is one that is of the form $dy/dt = f(y)$. In other words, the rate of change in some quantity y depends explicitly on the quantity y only. Thus, $dy/dt = 3y$ is autonomous, but $dy/dt = 3y + t$, for example, is not autonomous.

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