An instructional sequence for change of basis and eigentheory

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Instructional Design Theory: Realistic Mathematics Education

- Instructional design theory originating in the Netherlands (Freudenthal, 1991), Mathematics as a Human Activity

- Instructional sequences are designed to actively engage students in developing the mathematics by starting with students’ current ways of reasoning to build toward more formal, sophisticated mathematics

- Draws on instructional design heuristics such as Experientially Real Starting Points and Guided Reinvention

- Adopted for use in undergraduate mathematics courses such as Differential Equations (Rasmussen); Abstract Algebra (Larsen)

  Interested in these? Come grab a flyer about the TIMES project for more info!
Inquiry-Oriented Linear Algebra (IOLA)

Welcome to IOLA!
Developing Inquiry-Oriented Instructional Materials for Linear Algebra Instruction

About the Team »

What is IOLA?
Linear algebra is widely viewed as pivotal yet difficult for university students, and hence innovative instructional materials are essential. The goals of this project are to produce: (a) student materials composed of challenging and coherent task sequences that facilitate an inquiry-oriented approach to the teaching and learning of linear algebra; (b) instructional support materials for implementing the student materials; and (c) a prototype assessment instrument to measure student understanding of key linear algebra concepts. The project makes a needed contribution to the field by developing instructional materials that allow for active student engagement in the guided reinvention of key mathematical ideas. It also develops instructional support materials that convey the instructional designers’ intention without being overly prescriptive and that provide information about how students think and learn within the task sequences. The study partners mathematics education researchers and mathematicians to incorporate research on teaching and learning into effective pedagogical approaches at the undergraduate level. Sharing the instructional tasks and support materials is facilitated through the project website.

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Virginia Tech San Diego State University
Arizona State University
Florida State University
Inquiry-Oriented Linear Algebra (IOLA)

- **Main focus**: to make research-based task sequences more accessible to instructors interested in inquiry-oriented teaching of linear algebra

- **Typical classroom interactions**: Overviews types of classroom interactions that help foster a productive class environment (“typical day”)
  - Provides descriptions of various interactions (small group work, partner talk, whole class discussion, and lecture) that comprise an inquiry-oriented classroom
  - Gives detailed suggestions of how to foster productive whole class discussions

- **Three current stand-alone units**:
  - Unit 1: Span and linear independence (the Magic Carpet Ride sequence)
  - Unit 2: Matrices as linear transformations (the Italicizing N sequence)
  - Unit 3: Change of basis & eigentheory (the Blue and Black sequence)
Inquiry-Oriented Linear Algebra (IOLA)

For each task, the instructor support materials include three main components:

**Learning Goals and Rationale.** Addresses how the task contributes to meeting instructional goals and what kinds of thinking are meant to be evoked, leveraged, or challenged.

**Student Thinking.** Elaborates ways in which students might think about or approach tasks, answers/strategies they will likely develop, and difficulties they are likely to have.

**Implementation.** Includes specific suggestions for how to implement tasks in the classroom, what kinds of topics of debate might be most productive, and what types of student interaction teachers should anticipate.

Also contains: **Task Sheet** for students to use, **Homework Suggestions** for after the lesson, **Videos** from classes, and a **Discussion Board** to pose ideas or questions.
Unit 3 Overview  
“Blue and Black Task Sequence”

Supports students' reinvention of change of basis, eigentheory, and how they are related through diagonalization

- **Task 1:** Builds from students' experience with linear transformations in \( \mathbb{R}^2 \) to introduce them to the idea of stretch factors and stretch directions and how these can create a non-standard coordinate system for \( \mathbb{R}^2 \)

- **Task 2:** Has students create matrices that convert between the standard and non-standard coordinate systems and relate these to the stretching transformation of Task 1 to reinvent the equation \( A\mathbf{x} = PDP^{-1}\mathbf{x} \)

- **Task 3:** Builds from students’ experience with stretch factors and directions to create ways to determine eigenvalues and eigenvectors given various information about a transformation

- **Task 4:** Students work with examples in \( \mathbb{R}^3 \) to develop the characteristic equation as a solution technique, as well as connect eigentheory to their earlier work with change of basis through diagonalization
Task Setting:
A new linear transformation

Imagine a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ that has the following properties:

*In the direction along the line $y = -3x$, the transformation stretches all points by a factor of two.*

*In the direction along the line $y = x$, the transformation keeps all points fixed.*
**Task 1:**
The Stretching Task

1. Use the space on the right to sketch what should happen to the image shown on the left when it is stretched according to the transformation described above. You may use a combination of intuition or calculations, as well as any additional sketches below or on your group's whiteboard.

2. Determine what will happen to \( \begin{bmatrix} 0 \\ 2 \end{bmatrix} \) and to \( \begin{bmatrix} -2 \\ 2 \end{bmatrix} \) under this transformation. Use an initial estimate from your sketch in problem 1. Then try to do a calculation that will determine these locations more precisely.

3. Determine a matrix that allows you to calculate what happens under the transformation to any point on the plane. Use it to check your sketch or improve its accuracy.
Task 1:
The Stretching Task
A helpful coordinate system

Imagine a linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that has the following properties:

- *In the direction along the line $y = -3x$, the transformation stretches all points by a factor of two.*
- *In the direction along the line $y = x$, the transformation keeps all points fixed.*

Recreate the transformed $Z$, using the blue grid as a tool for sketching how the $Z$ gets transformed. Be able to explain in your own words how you used the blue grid to create the transformed $Z$. 
Task 2: The Blue and Black Task

Consider the following two coordinate systems of $\mathbb{R}^2$: the black coordinate system and the blue coordinate system.

1. Write the coordinates of each of the above points relative to both the blue and the black coordinate systems.

\begin{align*}
[a]_{\text{black}} &= [2] \\
[b]_{\text{black}} &= [-8] \\
[c]_{\text{black}} &= [-6] \\
[d]_{\text{black}} &= [4] \\
[e]_{\text{black}} &= [-11] \\
[f]_{\text{black}} &= [9]
\end{align*}

\begin{align*}
[a]_{\text{blue}} &= [3] \\
[b]_{\text{blue}} &= [-2] \\
[c]_{\text{blue}} &= [10] \\
[d]_{\text{blue}} &= [-2] \\
[e]_{\text{blue}} &= [-2] \\
[f]_{\text{blue}} &= [3]
\end{align*}
**Task 2:**
**The Blue and Black Task**

2. Determine a matrix that will:

   a. Rename points from the blue coordinate system as points in the black one.

   \[
   \begin{bmatrix}
   a & b \\
   c & d 
   \end{bmatrix}
   \begin{bmatrix}
   0 \\
   1 
   \end{bmatrix} = \begin{bmatrix}
   -3 \\
   9 
   \end{bmatrix}
   \]

   \[
   \begin{bmatrix}
   a & b \\
   c & d 
   \end{bmatrix}
   \begin{bmatrix}
   3 \\
   1 
   \end{bmatrix} = \begin{bmatrix}
   2 \\
   0 
   \end{bmatrix}
   \]

   \[
   A = \begin{bmatrix}
   1 & -1 \\
   1 & 3 
   \end{bmatrix}
   \]

   \[
   P = \begin{bmatrix}
   1 & -1 \\
   1 & 3 
   \end{bmatrix}
   \]

   

   \[
   3b = -3 \quad b = -1 \\
   3d = 9 \quad d = 3 \\
   3a + b = 2 \quad 3a = 3 \quad a = 1 \\
   3c + d = 6 \quad 3c = 3 \quad c = 1 
   \]

   b. Rename points from the black coordinate system as points in the blue one.

   \[
   \begin{bmatrix}
   a & b \\
   c & d 
   \end{bmatrix}
   \begin{bmatrix}
   0 \\
   1 
   \end{bmatrix} = \begin{bmatrix}
   1 \\
   1 
   \end{bmatrix}
   \]

   \[
   \begin{bmatrix}
   a & b \\
   c & d 
   \end{bmatrix}
   \begin{bmatrix}
   -8 \\
   0 
   \end{bmatrix} = \begin{bmatrix}
   -6 \\
   0 
   \end{bmatrix}
   \]

   \[
   \begin{bmatrix}
   3/4 & 1/4 \\
   -1/4 & 1/4 
   \end{bmatrix}
   \]

   \[
   \begin{bmatrix}
   3/4 & 1/4 \\
   -1/4 & 1/4 
   \end{bmatrix} = P^{-1}
   \]

   

   \[
   4b = 1 \quad b = 1/4 \\
   4d = 1 \quad d = 1/4 \\
   -8a = -6 \quad a = 3/4 \\
   -8c = 2 \quad c = -1/4 
   \]
Task 2: The Blue and Black Task

3. Recall the linear transformation from Task 1: vectors along the line $y = -3x$ get stretched by a factor of 2, and vectors along the line $y = x$ remain fixed.

Determine what happens to each of the vectors below under the transformation. Express the result in both the blue and the black coordinate systems. Describe your methods both graphically and with matrix equations.

a. $[\mathbf{x}_1]_{blue} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

b. $[\mathbf{x}_2]_{blue} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

c. $[\mathbf{x}_3]_{black} = \begin{bmatrix} -8 \\ -7 \end{bmatrix}$
There are multiple ways to get to the stretched version. You could, starting with the blue coordinates, double the y-coordinate \([\begin{bmatrix} 5 \\ 9 \end{bmatrix}]\) and then use \(P\) to convert to black coordinates system.

Or you could just go from \([\begin{bmatrix} 6 \\ 2 \end{bmatrix}]\) in blue and change it to black with \(P\) and then use \(A\) to find the stretched version.

\[
P \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 6 \end{bmatrix}
\]

Blue basis: \(x = x\),

\[
\begin{bmatrix} y & -3x \end{bmatrix}
\]

Stretch by factor of 2

So you can use

\[
B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}
\]

Transformation of point blue in black
Two ways for $[x]_{blue}$ to transform to $[y]_{black}$

$[x]_{black} \xrightarrow{A} A[x]_{black} = [y]_{black}$

$[x]_{blue} \xrightarrow{D} D[x]_{blue} = [y]_{blue}$

Top route:

$AP[x]_{blue} = [y]_{black}$

Bottom route:

$PD[x]_{blue} = [y]_{black}$

$AP = PD \Rightarrow A = PDP^{-1}$
In our example:

\[
\begin{bmatrix}
\frac{5}{4} & -\frac{1}{4} \\
-\frac{3}{4} & \frac{7}{4}
\end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix}
\frac{3}{4} & \frac{1}{4} \\
-\frac{1}{4} & 1/4 
\end{bmatrix}
\]

The two stretch directions

The two stretch factors

**Definition:**

An \( n \times n \) matrix \( A \) is called **diagonalizable** when it can be written as \( A = PDP^{-1} \) for a diagonal matrix \( D \) and an invertible matrix \( P \).
Task 3: Stretch Factors & Stretch Directions

1. The transformation defined by the matrix \( A = \begin{bmatrix} 1 & -8 \\ -4 & 5 \end{bmatrix} \) stretches images in \( \mathbb{R}^2 \) in the directions \( y = \frac{1}{2}x \) and \( y = -x \). Figure out the factor by which anything in the \( y = \frac{1}{2}x \) direction is stretched and the factor by which anything in the \( y = -x \) direction is stretched.

2. The transformation defined by the matrix \( B = \begin{bmatrix} -8 & 2 \\ -55 & 13 \end{bmatrix} \) stretches images in \( \mathbb{R}^2 \) in one direction by a factor of 3 and some other direction by a factor of 2. Figure out what direction gets stretched by a factor of 3 and what direction gets stretched by a factor of 2.

3. The transformation defined by the matrix \( C = \begin{bmatrix} 7 & -2 \\ 4 & 1 \end{bmatrix} \) stretches images in \( \mathbb{R}^2 \) in two directions. Find the directions and the factors by which it stretches in those directions.
Task 3: Stretch Factors & Stretch Directions

1. The transformation defined by the matrix \( A = \begin{bmatrix} 1 & -8 \\ -4 & 5 \end{bmatrix} \) stretches images in \( \mathbb{R}^2 \) in the directions \( y = \frac{1}{2}x \) and \( y = -x \). Figure out the factor by which anything in the \( y = \frac{1}{2}x \) direction is stretched and the factor by which anything in the \( y = -x \) direction is stretched.

\[
A = PDP^{-1} \\
A = \begin{bmatrix} 1 & -8 \\ -4 & 5 \end{bmatrix} \\
P = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\
P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \\
P'AP = D \\
D = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 9 \end{bmatrix}
\]

- 3 times along \( y = \frac{1}{2}x \)
- 0 times along \( y = -x \)
Task 3: Stretch Factors & Stretch Directions

2. The transformation defined by the matrix \( B = \begin{bmatrix} -8 & 2 \\ -55 & 13 \end{bmatrix} \) stretches images in \( \mathbb{R}^2 \) in one direction by a factor of 3 and some other direction by a factor of 2. Figure out what direction gets stretched by a factor of 3 and what direction gets stretched by a factor of 2.
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3. The transformation defined by the matrix \( C = \begin{bmatrix} 7 & -2 \\ 4 & 1 \end{bmatrix} \) stretches images in \( \mathbb{R}^2 \) in two directions.

Find the directions and the factors by which it stretches in those directions.
3. The transformation defined by the matrix $C = \begin{bmatrix} 7 & -2 \\ 4 & 1 \end{bmatrix}$ stretches images in $\mathbb{R}^2$ in two directions. Find the directions and the factors by which it stretches in those directions.

\[
\begin{bmatrix} 7 & -2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ka \\ kb \end{bmatrix}
\]

\[
7a - 2b = ka \\
4a + b = kb
\]

\[
7a - 2b = 0 \\
4a + b = 0
\]

\[
\frac{(7-k)}{-2} = \frac{4}{1-k} \\
7 - 8k + k^2 + 8 = 0
\]

\[
k^2 - 8k + 15 = 0 \\
(k-3)(k-5) = 0
\]

Could be leveraged here into the development of the characteristic polynomial
Task 4: Eigenvalues and Eigenvectors for a 3x3

Suppose $T: \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(x) = Ax$ and $A = \begin{bmatrix} 5 & 4 & -6 \\ 3 & 6 & -6 \\ 3 & 4 & -4 \end{bmatrix}$.

Answer the following questions regarding the transformation. Show your work.

1. If you know that the line $k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a stretch direction for the transformation $T$, what is the stretch factor (eigenvalue) associated with this stretch direction?

2. Given that 2 is a stretch factor (eigenvalue) for the transformation $T$, determine the set of all stretch directions (eigenvectors) associated with a stretch of 2.

3. From your work on questions 1 and 2, you know two different eigenvalues of the transformation $T$. Are there any others? If so, find (at least) one. If not, show there can’t be any more.
Theorem (The Diagonalization Theorem):

An $n \times n$ matrix $A$ is **diagonalizable** if and only if $A$ has $n$ linearly independent eigenvectors. In this case, the diagonal entries of $D$ are eigenvalues of $A$ that correspond, respectively, to the eigenvectors in $P$.

\[
A = PDP^{-1}
\]

\[
\begin{bmatrix}
5 & 4 & -6 \\
3 & 6 & -6 \\
3 & 4 & -4
\end{bmatrix} =
\begin{bmatrix}
1 & -4/3 & 2 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
3 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{bmatrix}P^{-1}
\]
Thanks for listening

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